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ABSTRACT

Focusing primarily on current problem areas in mathematics instruction, this third yearbook of the Finnish Association of Mathematics and Science Education Research contains five articles addressing four themes in mathematics education. The first theme is concerned with proportional reasoning in mathematics teaching and describes school test developments in the area. The next two articles deal with geometry teaching; the first outlines the aspects of instruction while the second discusses the role of computers in teaching. The third theme examines problem solving and the fourth describes learning difficulties of 13-year-old students in basic arithmetic operations. Biographic notes of the contributing authors are also included. (ML)

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Theory into practice 3

Pekka Kupari (Ed.)

MATHEMATICS EDUCATION RESEARCH IN FINLAND

Yearbook 1985

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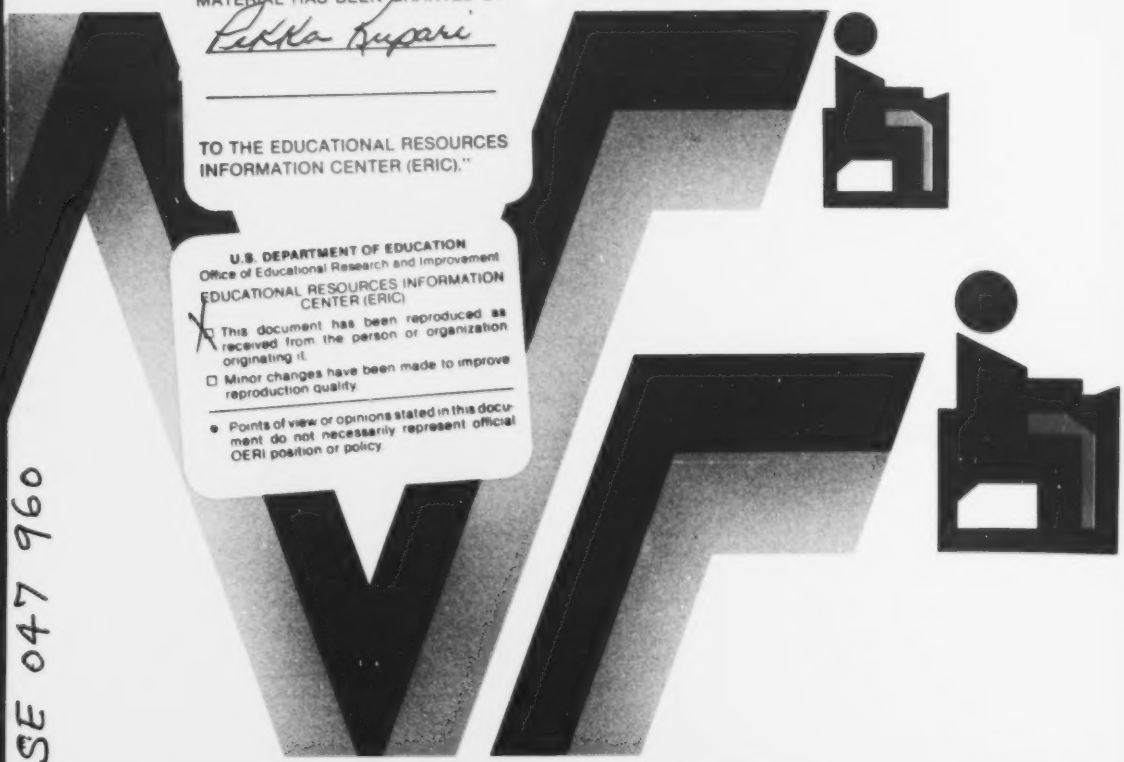

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The Finnish Association of Mathematics and Science Education Research

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MATHEMATICS EDUCATION RESEARCH IN FINLAND

YEARBOOK 1985

Edited by Pekka Kupari

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ABSTRACT

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The yearbook 1985 includes five articles of which two are written in German. The articles mainly present the authors' own research, some of them are lectures given in national seminars. The articles deal with four themes. The first is concerned with proportional reasoning in the teaching of mathematics in schools and description of school test developments. The next two articles deal with geometry teaching: the first outlines the perspectives of instruction and the second discusses the role of computers in teaching. The third theme is problem solving and its more extensive understanding. The last article describes the learning difficulties of 13-year-olds in basic arithmetic operations as well as one approach to helping pupils.

Descriptors: mathematics, reasoning, geometry, computer, problem solving, learning difficulty

PREFACE

The Finnish Association of Mathematics and Science Research has been active for little over three years. One of its most important tasks has been to disseminate information on research in the field also outside Finland. The research association has published its yearbook for these purposes and the present yearbook is the third. The two previous yearbooks have also dealt with research on mathematics teaching.

The new school laws enforced in fall 1985 represented fairly great changes in the teaching of mathematics in the comprehensive school. On the upper level sets (streaming) was abandoned. We used to have two sets on the 7th grade and three on grades 8 and 9. A change from administrative to within teaching group differentiation was made. Instruction is now based on a syllabus common to all pupils and possible re-grouping of pupils is done on the basis of the time resource quota-system. The system simply means that municipalities and schools can themselves allocate teaching resources according to local needs. Curricular changes can best be described by the objectives of the 1985 mathematics syllabus, which emphasizes more than before e.g. the development of pupils' thinking and problem solving skills as well as the development of their creativity.

What kinds of pressures and needs are directed to the development and reform of teaching? How could the problematic geometry teaching be made more vigorous and uniform? What have computers and technology to offer? Can new ways be found for the mastery of learning difficulties? An attempt is made to answer and discuss these current problem areas in the articles of this yearbook.

The yearbook's appearance has changed. This results in the fact that the Institute for Educational Research (University of Jyväskylä) responsible for the layout and printing of the yearbook has renewed its report series. In years to come the yearbook will be published in this form.

Pekka Kupari

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RATIONAL AND EMPIRICAL ANALYSIS FOR THE MEASUREMENT AND INSTRUCTION OF PROPORTIONAL REASONING IN SCHOOL

Tapio Keranto

ABSTRACT

This study is the first part of a research project investigating the development and instruction of proportional reasoning. The main task of the study was to construct and try out classroom tests of proportional reasoning which are linked to the teaching of ratio and proportion in school, and which would help to measure each pupil's level of performance and strategies in a relatively easy and reliable way. The basic idea in the construction of the tests was to compose, on the basis of previous studies involving rational task analysis and oral interviews, certain types of item sets. Basically the idea was that the pupil should not be able to solve the whole of a given combination of items using the strategies and counting skills that are logically characteristic of a lower item combination. It was found that the test types Juice I ($o_1 : w_1, \sum o_2 : w_2$), Juice II ($o_1 : w_1 = o_2 : x$, $o_2 = m/n \cdot o_1$ & $x = m/n \cdot w$), Juice III ($o_1 : w_1 = x : y$, & $x+y=c$, $x = m/n \cdot o_1$ & $y = m/n \cdot w_1$) could be seen to form hierarchies in respect of certain item combinations. The development of proportional reasoning in the primary school was found to be very consistent. The results were seen to have implications for the teaching of proportional reasoning and ratio in school and for syllabus design.

INTRODUCTION

This report is based on the results of the first part of a research project investigating the development and instruction of proportional reasoning (Keranto 1985 b).

The main task of the first part of the project was to construct and try out classroom tests of proportional reasoning linked to the teaching of ratio and proportion in school, and would help to measure each pupil's level of performance and "strategies in a relatively easy and reliable way".

Previous studies involving oral interviews formed the theoretical and empirical basis and starting point for the construction of the tests. On the basis of these studies, we have quite detailed information on the possible solutions of certain proportional problems - rational analysis - and about the pupils' solutions to these problems - empirical analysis - on each level of cognitive development (Noelting 1980 a, b, Keranto 1985 a, Lehtonen & Lungberg 1985, Nurmi et al. 1984). The basic idea in the construction of the tests was to compose, on the basis of the above mentioned studies, certain types of item combinations. From the answer patterns of these combinations, an attempt was made to infer the strategies used and each pupil's level of performance.

The question how the item combinations for each test should be chosen so that they would prevent the pupil from solving the whole of a given combination by using "immature" strategies characteristic of lower item combinations, the existence of which has been proved on the basis of logical analysis, became the central problem in the construction of these tests.

Similarly there was originally a problem of how to choose the different test types so that they would have clear connections with the teaching of the idea of ratio and proportional reasoning in school, and with each other. The success of this attempt will be discussed briefly in the following.

PROPORTIONAL REASONING

The objective of the study was limited to concern proportional

reasoning in the context of liquid mixtures. That is to say, it was the pupil's task to conclude from the ratios of the components in two liquid mixtures, which one of the liquid mixtures was "stronger", or if they were "equally strong", and how the liquid mixtures could be made "equally strong". In mathematical terms, the question is of the proportionality of two compositions or of putting them into proportion with each other. Two combinations are in proportion with each other, if the ratios of the components are the same in both compositions. The relative strength of the mixtures can be proved in general outline by studying either the "internal" ratios of both mixtures (for instance, $juice_1$:$water_1$ and $juice_2$:$water_2$) or the "external" ratios between the mixtures ($juice_1$:$juice_2$ and $water_1$:$water_2$). The problem-solving strategies can be classified as follows: (see Noelting 1980 a, Keranto 1985 a):

A. The within strategies aim at the reduction of relative comparison to additive comparison of the values of the ratios. This takes place via a division operation. The values of the ratios can be presented in the form of integers, fractions, decimals or percentages. In Freudenthal's terms, the logical status of the task is "dropped" in this way (cf. Freudenthal 1983, 179-181).

B. The between strategies aim at the follow-up of the covariation of the "juices" (and the "waters") by the help of multiplied operations. In fact, here we have in the background the idea that it is possible to conclude the equivalence (or inequivalence) of the ratios without formally counting the values of the ratios. The use of this strategy is connected with the formal calculus of the extension and cancellation of fractions in school.

In principle, the mixture tasks can be solved by using the above mentioned strategies together or separately. As regards this study, it is essential that it is possible to describe the development of proportional reasoning in the context of liquid

mixtures by means of a logical hierarchy. The existence of this hierarchy has been proved in its general outline also experimentally (Noelting 1980 a, b, Keranto 1985 a, Nurmi et al. 1984).

The attainment of a higher level in this hierarchy requires both quantitative and qualitative changes in the components of the pupil's cognitive development; as these components were classified the non-numerical schemata (the idea of relative comparison), solution strategies and procedural skills. Psychologically, one of the basic prerequisites for proportional reasoning is the development of the idea or the schema of relative comparison. At a certain stage the pupil should be able to make a difference between relative comparison and absolute comparison. For instance, the ratios 1:2 and 2:4 have the same value, although the corresponding liquid mixtures differ from each other as regards the quantities. In practice the above mentioned situation means that the question is of "equally strong" liquid mixtures or that while the quantity of the juices is doubled the quantity of the waters should also be doubled. According to the study, the item combinations connected with the understanding of the equivalence classes $\{j:w \mid j:w=1:1\}$ and $\{j:w \mid j:w=j_o:w_o$, where $j_o \neq w_o\}$ form a kind of "watershed" between the problem types to be solved by "intuitive operations" and the problem types to be solved by "formal operations". In the tasks related to the equivalence classes in question, also the skills connected with the multiplication and division operations are needed for the first time. In the problems of formal operations, at the next level of difficulty, traction, decimal and percentage skills will also come for real use and need. So the level of difficulty in each three cognitive components rises with transition to the upper levels of the hierarchy.

But the above data is not yet sufficient if we want to construct good classroom tests. What should be added to the above analysis, will be discussed briefly in the following.

THE CONSTRUCTION AND CONTENT OF CLASSROOM TESTS IN PROPORTIONAL REASONING

The basic starting point for the construction of classroom tests was Noelting's "Orange Juice Experiment" (Noelting 1980 a, Keranto 1985 a). The choice of suitable item combinations was problematic in the construction of classroom tests corresponding to Noelting's test. How to construct such classroom tests, based on multiple choice that would make it possible to define reliably the pupil's level of performance in proportional reasoning and the strategies he used on the basis of his answers? The analysis of the results of the experimental measurements performed indicated that it is no use studying the items one by one, but by convenient groups or combinations. To be sure, it is certainly not the first time that this happens in a research process!? It became evident that the item combinations should be constructed so that the pupil could not solve all the items in a certain combination by such "immature" strategies that are logically characteristic of a lower item set. It appeared that this point has not been sufficiently considered and taken into account in Noelting's test. In fact the pupil can in principle reach the two higher levels of "intuitive operations" in the test above (the levels IB and IC) by using the hierarchically lower level (the level IA) unidimensional comparison of waters (cf. Noelting 1980 a, table 1, table 2, table 3, items 1, 3, 5 and 8, 10, 13, p. 220, 227, 228). This is probably not a great problem in a study based on interviews and small group work, but it should be taken into account in the construction of a classroom test.

Figure 1 explains in detail the meaning of what is said above (see Appendix 1). In the same way, it explains the structure of the test type JUICE I, which corresponds to Noelting's test, the item combinations used and the strategies leading to both right and wrong result patterns. Figure 1 also shows what kind of items should be added to the JUICE I test in future so that the pupil should not even in theory be able to solve each

combination by such strategies that in principle are not sufficient for the complete solution of the item combination.

The test types JUICE II and JUICE III were also constructed in addition to the test type JUICE I. The purpose was to study the validity of JUICE I and the connections of these three tests with school mathematics teaching.

In the test type JUICE I the pupil had to compare the relative strength of two liquid mixtures with each other when all the components of the mixtures (juices, waters) were known. The structure of the problem was of the form $j_1:w_1 \lesseqgtr j_2:w_2$. The hierarchy referred to was established by appropriately varying the number of components.

In the test type JUICE II the quantity of the juices and waters to be poured into one jug (jug A) and the quantity of juices (or waters) to be poured into the other jug (jug B) was known. It was the pupil's task to conclude how many glasses of water (in addition to the juices) should be poured into jug B, so that there will be an "equally strong" mixture in both jugs. The structure of the problem was of the form $j_1:w_1 = j_2:x$ (or = $x:w_2$), where $j_2 = m/n \cdot j_1$ and $x = m/n \cdot w_1$. From a logical point of view, the solution of the problems in the test type JUICE II requires mastery of the equivalence classes $\{j:w \mid j:w=1:1\}$ and $\{j:w \mid j:w=j_0:w_0$, where $j_0 \neq w_0\}$.

Thus it seems probable that those pupils who master the combinations IIA and IIB of "concrete operations" in the test type JUICE I, will also master the corresponding problems in the test type JUICE II, and vice versa. This will be discussed further in connection with the experimental part.

In the test type JUICE III the quantity of the juices and waters to be poured into one jug (jug A) and the total amount of the mixture (C liters) to be poured into the other jug (jug B) was known. It was the pupil's task to divide the total amount in the

ratio $j_1:w_1$. The structure of the problem was of the form $j_1:w_1 = x:y$, where $x=m/n \cdot j_1$ and $y=m/n \cdot w_1$ and $x+y=C$.

On the basis of a rational task analysis, the setting of the problems in the test type JUICE III hints at the use of a "try and see" and/or a "measurement" strategy. If the pupil uses the "try and see" strategy it is to be expected that the pupil, who has attained the highest level IC of the "intuitive operations" in the JUICE I test, in theory could solve even problems related to the equivalence class $\{j:w \mid j:w=j_0:w_0$, where $j_0 \neq w_0\}$ in the test type JUICE III. Thus the test type JUICE III may prove to be the easiest of all! We should bear in mind, however, that in the problems of the test JUICE III the solutions take place in the head, although all the knowledge needed is in sight on the overhead transparency. The pupils' memory is put to a test during the process, and this viewpoint has not yet been taken into account in this study (cf. Case 1980, Keranto 1984, 1985 a). On the other hand it should be pointed out that the given values needed in the test were natural numbers and the ratios used the simplest possible. Thus the understanding of the schema of relative comparison became crucial in the solution of the problems, as well as suitable solution strategies. The realization of these strategies did not require more advanced fraction and decimal skills. It should be pointed out that the object of interest was, as we will see in the experimental part, the spontaneous strategies of primary school pupils in the test types JUICE I, II and JUICE III which require proportional reasoning. On the basis of the above analysis of the test types JUICE I, II and III, it becomes apparent that the types JUICE II and JUICE III have clear connections with school mathematics. The test type JUICE II is connected with the definition of one member of the proportion, when the other members of the proportion are unknown. The test type JUICE III is connected with the teaching of the "distribution in a given proportion" -problems in school.

EXPERIMENTAL OBSERVATIONS

The main problems of the experimental part of the study were presented in the form of questions as follows:

A. How consistently did the pupils on each grade (grades 1 to 6) answer the item combinations of the test type JUICE I?
B. What kinds of strategies, derived from the result patterns of the item combinations in the tests JUICE I and JUICE II, did the primary school pupils use in their solutions?
C. How and to what extent could the item combinations of each test type be arranged so as to form hierarchies?
D. To what extent were the performance levels in each test interdependent?
E. To what extent does proportional reasoning develope during the first six school years?

Method and Subjects

The measurements involved in the experimental part of the study were performed as a classroom test consisting of two parts in the training school of the Department of Teacher Training in Hämeenlinna in January 1985.

The 331 pupils of the lower grades in the above mentioned comprehensive school formed the target group of the study. 92 % of these took part in both tests. The number of pupils in each grade level, the number of pupils participating in the tests, and the loss in each grade can be seen in the Table 1 (Appendix 2).

The JUICE I test, which originally consisted of 30 items, was the first test to be presented, and after that (usually after about a week's interval) the JUICE II and III tests were presented in one go (Appendix 3). The time used for the test on each grade varied from 60 minutes (6th grade) to two lessons

(2 45 min/1st grade). Other grades did the tests in the above mentioned interval so that the time spent on the test shortened systematically. In all the testing took 28 lessons. The author of this study conducted the testing by himself in order to improve the stability of the tests in this way.

There was a instruction with models at the beginning of each test (Appendix 4). Each pupil had been presented with an answer sheet, and the pupils were given explanations on how to use it in connection with the introduction and model solution of each test type (Appendix 5). In addition to this the central question of each test type was written on the blackboard. In connection with the JUICE I test instructions on when case A ("the mixture in Jug A will be stronger"), when case B ("the mixture in jug B will be stronger") or case E ("equally strong") should be ringed, were written on the blackboard.

Each problem was presented with on overhead projector on transparencies. The pupils could only use a pencil and a ruler. If the pupil made a mistake he was advised to draw a tick on the "wrong" performance and ring the other alternative (the JUICE I test) or to mark number(s) (the tests JUICE II and III). When the task was completed the pupil had to place the pencil on the desk and move the ruler to the following line on the answer sheet. When all the pupils had placed their pencils on the desk and moved their rulers to the following problem, the following problem was taken up. On the whole the system worked quite satisfactorily; a further indication of this is the fact that

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1) In the Finnish school system, the comprehensive school consists of:

- the lower grades: grades 1 to 6, for pupils from 7 to 12 years old. In this report, the term "primary school" is used to refer to the lower grades of the comprehensive school
- the upper grades: grades 7 to 9, for pupils from 13 to 15 years old.

only one performance had to be disqualified on account of incorrect filling of the answer sheet.

Classroom Tests JUICE I, JUICE II and JUICE III

As noted earlier, the JUICE I test was developed on the basis of Noelting's Orange Juice test, and it consisted originally of 30 items. Later on 10 items were eliminated from the test on the basis of an item-related hierarchical analysis. Thus our JUICE I test has 20 items, the construction and item combinations of which were introduced already in Figure 1 (Appendix 1). The sequence of the problems in the JUICE I test chosen to the final analysis is shown in Appendix 3.

The JUICE II test was a modification of the JUICE I test and it consisted of six problems. Out of these, three problems measured the understanding of the equivalence class $\{j:w \mid j:w=1:1\}$ and the other three the understanding of the equivalence class $\{j:w \mid j:w=j_o:w_o$, where $j_o \neq w_o\}$ (Appendix 3, compare the problems 12-17 in the JUICE I test).

The JUICE III test, again, was a modification of the JUICE II test and it consisted of eight problems (Appendix 3). Out of these, four problems measured the understanding of the equivalence class connected with the ratio 1:1 and the other four problems the understanding of the equivalence class of an "arbitrary" ratio r:w, j=w (compare the corresponding problems in the JUICE II test).

The Results

The aim is to answer in three parts the main problems A - E of the experimental part of the study. The results connected with problems A & B will be presented first, then the results con-

nected with problems C & D and finally an attempt will be made to give an answer to problem E.

How consistently did the primary school pupils answer the item combinations of the JUICE I test and what kinds of strategies did they use?

Let us take a look at the following Table 2, which shows the percentages of the result patterns concerning the item sets IA-III on each grade (cf. also Figure 1, Appendix 1). In the table referred to, the column "OTHERS" stands for "inconsistent" behaviour, that is to say, the performance pattern of the pupil differs partly from the result pattern based on the use of a certain strategy.

Table 2. The percentages of the result patterns of the item combinations IA-III in the JUICE I test on grades 1 to 6.

Item combination / Result pattern	IA (1&2)			IB (3&4)				IC_1 (5-8)					IC_2 (9-11)				
	AB	EE	sp	AA	EE	BB	sp	ABAA	AABB	EABB	AAAB	sp	ABA	ABB	BAA	EEE	sp
1. (N=53)	91	0	9	81	0	9	10	30	9	0	0	61	64	13	0	0	23
2. (N=53)	92	0	8	83	4	0	13	53	8	0	4	35	68	13	0	0	19
3. (N=54)	87	3	10	91	0	0	9	63	4	0	0	33	72	4	0	0	24
4. (N=52)	96	0	4	88	2	4	6	69	0	0	4	27	71	2	0	0	27
5. (N=57)	92	0	8	96	0	2	2	75	2	0	2	21	75	2	0	0	23
6. (N=37)	100	0	0	100	0	0	0	86	0	0	0	14	73	0	0	0	27

sp = some other result pattern

Item combination / Result pattern	IIA (12-14)				IIB (15-17)					III (18-20)				
	EEE	BBA	AAB	sp	EEE	BBA	AAB	AAA	sp	ABB	ABA	BAB	EEE	sp
1. (N=53)	36	9	0	55	4	15	2	15	64	0	13	0	26	61
2. (N=53)	55	8	2	35	4	9	4	43	40	4	13	4	23	56
3. (N=54)	72	15	0	13	19	9	2	28	42	4	9	4	44	39
4. (N=52)	81	4	0	15	8	8	0	35	49	2	10	2	56	30
5. (N=57)	88	4	2	6	23	2	0	32	43	5	2	2	46	45
6. (N=37)	84	0	2	14	54	3	5	11	27	14	3	0	54	29

On the basis of Table 2 it can be stated that, generally speaking, the consistency of the answers to each item combination grew with the transitions to upper grades. Similarly it can be noted that the easier the item combinations in question are, the more consistently the pupils will answer them. In the item sets IA, IB, IC and IIA the percentages of consistency were quite high, 70 % - 100 %, on the upper grades. This indicates that particularly the pupils of the upper grades (grades 3 to 6) use the same strategy quite constantly throughout the whole item combination. The character of these strategies will be discussed in the following.

Let us bear in mind the result patterns connected with the item combinations IA-III in Figure 1 (Appendix 1) and examine the strategies to be inferred from the result patterns of the item IC-III in Table 2. First of all it can be seen in the result patterns of the item combination IC_2 that the "most popular", consistently used strategy leading to an incorrect answer combination is either "the comparison of the quantities" or the "unidimensional comparison of the juices" (result pattern ABB). It is evident that the pupils of the 1st and 2nd grade in particular use one or the other of these strategies leading to an incorrect result pattern. On the upper grades the consistent use of these strategies is of little importance (4 % - 0 %). It is significant that consistent, unidimensional comparison of the waters (pattern BAA) does not occur on any grade. In other words, the pupils seem to pay attention to the juices rather than to the waters in the problems of the level IC_2 referred to. In the item combinations IIA and IIB, a similar trend can be seen as in the item combinations IC_2. The unidimensional comparison of the juices (and, to a certain extent, of the waters) decreases with the transition to upper grades (the patterns BBA and AAB). The consistent strategy (in this item combination leading to the incorrect pattern AAA) used in the item combination IIB is "count the between or within difference". Only on the sixth grade does this strategy give way to multiplied operations (pattern EEE). The most favoured strategy in the item

combination III is also "count the between or within difference" (pattern EEE). The pupils of the upper grades (grades 3 to 6) in particular seem to use this strategy, which, consequently, leads to an incorrect pattern quite consistently (cf. Lehtonen & Lundberg 1985, Nurmi et al. 1984).

The following table gives a more detailed picture of how the use of the "difference" strategy, which leads to an incorrect pattern, grows quite consistently in the combinations IIB and III during the first six school years.

Table 3. The percentages (%) of incorrect answers, based on the use of the "difference" strategy, of all the incorrect answers in the combinations IIB and III of the Juice I test.

Item combination / Result pattern	IIB				III		
	1:2 vs 2:4		1:3 vs 2:6		2:3 vs 1:2	2:3 vs 3:4	4:3 vs 3:2
	A	B	A	B	E	E	E
1.	61	39	63	37	77	72	52
2.	80	20	79	21	86	79	55
3.	76	24	74	26	81	85	67
4.	83	17	80	20	88	95	81
5.	88	12	90	10	91	91	83
6.	92	8	92	8	96	96	91

On the basis of Table 3, percentages of "difference" strategies leading to an incorrect result in the combinations IIB and III of all the strategies leading to incorrect results grows systematically with transition to upper grades. Correspondingly, the results based on unidimensional strategy decrease systematically with transition from the lower to the upper grades. The results are logical and agree with the previous theoretical and empirical studies (Noelting 1980 a, b, Keranto 1985 a, Lehtonen & Lundberg 1985, Nurmi et al. 1984). The observations referred to can be anticipated also on the basis of Piaget's theory at a more general level (Piaget et al. 1960, Piaget 1970, Piaget 1977).

The following table contributes to the results achieved and shows the percentages of the incorrect answers 0-5 of all the incorrect answers to the problems associated with the understanding of the equivalence class $\{j:w \mid j:w=j_o:w_o$, where $j_o \neq w_o\}$ in the JUICE II test.

Table 4. The percentages (%) of incorrect answers 0-5 of all the incorrect answers to problems 4-6 in the JUICE II test.

	item 4							item 5								item 6							
	1:2 vs 2:?							1:3 vs 2:?								4:2 vs 2:?							
answer	0	1	2	3	5	s	a/n	0	1	2	3	4	5	s	a/n	0	2	3	4	5	s	a/n
1.	0	17	23	55	2	3	53/53	2	2	17	26	43	0	10	53/53	18	32	5	25	2	18	44/53
2.	0	11	13	72	0	4	53/53	0	2	12	8	69	4	5	49/53	40	17	7	26	2	8	42/53
3.	0	10	20	69	0	1	49/54	0	0	22	4	74	0	0	46/54	40	8	0	43	2	7	40/54
4.	0	8	8	84	0	0	47/52	0	0	6	4	83	4	3	47/52	62	6	3	26	0	3	34/52
5.	0	6	4	88	2	0	48/57	0	2	6	2	80	10	0	41/57	38	0	8	38	0	16	13/57
6.	0	10	0	90	0	0	21/37	0	0	12	6	70	12	0	14/37	42	29	0	29	0	0	7/37

s = some other incorrect answer than 0-5

a/n = the number of incorrect answers per number of subjects

It can be seen from the percentages of the incorrect answers to problems 4-6 that the pupils have relatively most often arrived at such answers where the difference between the waters and juices to be poured into Jug A is the same as the difference between the waters and juices to be poured into jug B (as an answer 3/item 4, as an answer 4/item 5, and as an answer 0/item 6). This indicates very clearly the use of the "difference" strategy in the solutions of the problems in question in grades 1-6. The result illustrates the obviously very close connection between the performances in the JUICE I and JUICE II tests.

<u>To what extent are the item combinations hierarchical and related to each other?</u>

The performance levels associated with the item combinations of

the tests in question can in principle be defined in several different ways. The basic principle was that the performance levels associated with the corresponding item combinations of the tests referred to would correspond with each other as well as possible also as regards the number of right solutions. The criteria for the performance levels of each test is presented in Figure 2 (Appendix 6).

As it appears in Appendix 6 there were originally three criteria (A) , (B) and (C), which were connected with the performances in the JUICE test. The values of hierarchy CR, MMR and PPR corresponding to these criteria are presented in the following table (see Guttman 1944), White & Saltz 1957). The recommendation for a "perfect" hierarchy is CR>0.90, MMR<0.80 and PPR>0.70.

Table 5. The values of hierarchy CR, MMR and PPR connected with the performances of item combinations IA-III in the JUICE I test according to the three different criteria (A), (B) and (C).

	Criteria (A)			Criteria (B)			Criteria (C)		
class	CR	MMR	PPR	CR	MMR	PPR	CR	MMR	PPR
1.	.87	.78	.41	.90	.74	.62	.92	.76	.67
2.	.87	.77	.43	.91	.83	.47	.94	.85	.60
3.	.87	.80	.35	.93	.88	.42	.95	.77	.78
4.	.87	.85	.13	.93	.91	.22	.97	.93	.57
5.	.89	.86	.21	.97	.92	.63	.98	.75	.92
6.	.79	.77	.09	.97	.82	.83	.99	.83	.94

It can be noted that the criteria (C) best realizes the above mentioned recommendations and it also corresponds with the performance levels IA-III of this study. The result indicates that the performance level of the pupil can be defined quite reliably by means of the answer patterns of the item combinations IA-III. In addition to this the performance levels IA, IB, IC, IIA, IIB and III connected with the item combinations can be arranged so as to form hierarchies on the basis of what has been stated

above. How and to what extent were the performances of the JUICE II and JUICE III tests, to be arranged so as to form hierarchies?

It was discovered that there was not a single pupil in the JUICE II test who solved correctly at least two items out of three connected with the arbitrary equivalence class, while he had solved correctly less than two items out of three connected with the equivalence class 1:1. In the JUICE III test, 4 pupils were found (out of 306 pupils) who had answered correctly to three items out of four logically on a more demanding level, while they failed in the corresponding items which were logically on a lower level. On the basis of this observation and the criteria, it can be stated that also the JUICE II and JUICE III tests can be arranged so as to form hierarchies.

To what extent, on the basis of a correlative analysis, are the performance levels of the tests in question interdependent?

The correlations were counted as Pearson's correlation coefficients by attaching the numbers 1 (I), 2 (IIA) and 3 (IIB or III) and 1 (I), 2 (IIA), 3 (IIB) and 4 (IIB) to the performance levels. The following table presents in detail the correlations referred to.

Tabel 6. The correlations of the performance levels of the JUICE I, JUICE II and JUICE III tests as Pearson's correlation coefficients

class	JUICE I & II	JUICE I & III	JUICE II & III	
1.	.57***	.23	.22	(N=53)
2.	.30*	.19	.27*	(N=53)
3.	.41**	.47***	.04	(N=54)
4.	.63***	.52***	.48***	(N=52)
5.	.38**	.51***	.50***	(N=57)
6.	.53***	.49**	.57***	(N=37)

* = p<.05 , ** = p<.01 , *** = p<.001

As it could be anticipated on the basis of the previous analysis, the performance levels of the JUICE I and JUICE II tests in particular correlate clearly on all grades. The same holds true as regards the JUICE I and JUICE III tests on the upper grades 3 to 6. The performance levels of the JUICE II and JUICE III tests correlate highly on grades 4 to 6. The results presented here are parallel with logical analysis and appear to be also otherwise logical. It is evident that by the help of the JUICE I test it is possible to anticipate the success in the JUICE II and JUICE III tests and also vice versa.

<u>To what extent does proportional reasoning develop during the first six school years?</u>

To begin with, let us examine the development of proportional reasoning in the problems of the JUICE I type. The following table shows in detail the frequencies and percentages of the performance levels on each grade in the JUICE I test.

Table 7. The frequencies and percentages (%) of the performance levels IA-III in the JUICE I test.

		Performance levels						
		IA	IB	IC	IIA	IIB	III	Σ
class	1.	6	11	9	25	2	0	53
		11	21	17	47	4	0	100 %
	2.	2	10	11	28	2	0	53
		4	19	21	52	4	0	100 %
	3.	1	7	7	29	10	0	54
		2	13	13	53	19	0	100 %
	4.	2	1	2	43	4	0	52
		4	2	4	82	8	0	100 %
	5.	1	1	2	39	13	1	57
		2	2	4	68	22	2	100 %
	6.	0	1	2	14	15	5	37
		0	3	5	38	40	14	100 %

On the basis of the above table, the percentages of performance levels on each grade form a clear trend. From the first to the fourth grade there is a slow transition to the IIA level so that on the 4th grade most of the pupils have attained the level of "lower concrete operations" in proportional reasoning. The transition to the IIB level ("higher concrete operations") seems actually to begin on the fifth grade. Only on the 6th grade just over a half of the pupils have attained at least the IIB level.

It is to be anticipated on the basis of the previous correlation analysis that there is in the JUICE II a similar cross sectional result as in the JUICE I test. In fact, it is exactly so as can be seen in the table below. In examining Table 8 it should be taken into account that the performance level IIB of the JUICE II test corresponds with the performance level IIB of the JUICE I test both logically and as regards the content.

Table 8. The frequencies and percentages (%) of the performance levels I-IIB_2 in the JUICE II test.

	Performance levels				
class	I	IIA	IIB_1	IIB_2	Σ
1.	24	29	0	0	53
	45	55	0	0	100 %
2.	11	40	2	0	53
	21	75	4	0	100 %
3.	10	37	3	4	54
	19	68	6	7	100 %
4.	3	44	1	4	52
	6	85	2	7	100 %
5.	5	35	9	8	57
	9	61	16	14	100 %
6.	0	14	10	13	37
	0	38	27	35	100 %

It can be noted that the "higher concrete operations" measured by the JUICE II test are attained quite slowly in the same way as in the JUICE I test. The level of attainment of the "lower concrete operations" is almost 100 per cent already on grade 4. These results have certain implications on the syllabus plan and on the teaching of proportional reasoning in school. The matter will be taken up again later on.

How then, measured by the JUICE III test, does proportional reasoning develop? The following table gives us information concerning this point.

Table 9. The frequencies and percentages (%) of the performance levels I-IIB$_2$ in the JUICE III test.

class	Performance levels I	IIA	IIB$_1$	IIB$_2$	Σ
1.	20	27	5	1	53
	38	51	9	2	100 %
2.	6	26	16	5	53
	11	49	30	10	100 %
3.	9	10	26	9	54
	17	19	47	17	100 %
4.	4	14	20	14	52
	8	27	38	27	100 %
5.	6	13	19	19	57
	11	23	33	33	100 %
6.	1	6	8	22	37
	3	16	22	59	100 %

On the basis of logical analysis, it could be anticipated that the higher performance levels IIA, IIB and IIB will be attained earlier than the corresponding performance levels in the JUICE I and JUICE II tests. In fact, it becomes evident that already on the second grade nearly 90 % of the pupils have attained at least the IIA level. On the third grade clearly more

than a half (64 %) of the pupils have an idea of the performances of the IIB level. There is no significant development during the fourth and fifth grades, and the corresponding percentages are 65 % and 66 %. On the sixth grade already 81 % of the pupils have attained at least the "transitional stage" IIB_1. For the first time, more than 50 % of the pupils (59 %) have now reached the IIB_2 level ($\hat{=}$IIB level/JUICE I). In this respect the JUICE III and JUICE I tests are on the same level, whereas the JUICE II test proves to be more difficult. On the other hand, it should be remembered, that although an attempt was made to define the criteria of the performance levels in question so that they would correspond with each other, there are still some differences. This contributes to the differences in the attainment of the performance levels and to their comparability.

IMPLICATIONS OF THE PRESENT STUDY ON THE TEACHING OF PROPORTIONAL REASONING IN SCHOOL AND PROSPECTS FOR FUTURE RESEARCH

In the mathematics curriculum the instructional goals for the idea of ratio and proportional reasoning seem according to this study partly too "light", partly they have been completely neglected, and partly they appear to be too demanding. In fact the developing of proportional reasoning could be started already on the lower grades. Each pupil could perform tasks which develope proportional reasoning and the idea of ratio and correspond with his own level of development. These problems could be presented at an oral-manipulative-visual level maintaining the connection with real life situations. In this respect the curriculum does not give any clear indications (cf. the Curriculum of Mathematics in the Comprehensive School 1982). On the other hand, in the curriculum the counting of the value of the ratio and of the percentage number are presented as the objective on the sixth grade. The term "proportion" does not occur until in the syllabus of the upper grades of the comprehensive school. According to this study, a significant part (40-59 %) of

the pupils of the sixth grade had not yet attained the level of understanding the equivalence class of an arbitrary ratio in the context of mixture problems. If we come to the assumption that operations on the symbolical level in mathematics become understandable and meaningful via the oral-manipulative-visual representations of real life, the situation is problematic. It seems that the meaning and appropriateness of counting the percentage can be quite questionable for some of the pupils on the sixth grade. In other words, it can be quite useless and even harmful to hasten to the written definitions of the value of the ratio and of the percentage before these activities have sense and meaning for the pupil.

It would be very important and interesting to examine how this kind of a pupil for whom the mathematical symbols have no basis in real life and whose cognitive facilities to understand symbolical operations are poor, experiences and perceives mathematics and its formal calculus. Could it be actually so, as Freudenthal indicates, that the "buds" of proportional reasoning and the idea of ratio are stunted in their growth by manipulations not connected with real life situations (cf. Freudenthal 1983, 206-209). If this is true, it may have serious consequences as well as further studies in mathematics and the attitudes towards mathematics studies are concerned and, later on, attitudes towards the social meaning of mathematics (see Karjalainen 1982, Keranto 1983, Salonen et al. 1982).

All that has been said above speaks for the increase of genuine real life practical and problem-solving exercises and for their connection with the symbolical operations of mathematics (cf. Leino & Norlamo 1980). This relates also to mathematics teaching in the upper grades of the comprehensive school and in the upper secondary shcool, where the psychological side of mathematics teaching and learning obviously has not yet been sufficiently taken into account. In other words we should ensure that also here the transition to the level of operating with symbols should not be made before the pupil can connect the manipulative

-visual-verbal presentations with real life situations at certain problem levels of proportional reasoning. For instance, there is no need - and it is obviously not even reasonable - to wait for the teaching of the "distribution in a given proportion" up to the 8th school year. It was found that at the real life oral-manipulative-pictorial level a great number of the pupils on the second grade of the comprehensive school already had facilities for the solving of the problems of a certain level in the problem type in question! Why should we not utilize these abilities and prepare symbol-level activities with the handling of proportion earlier than it is proposed in the curriculum?

There are some difficult problems in realizing the above mentioned in school mathematics teaching in a larger scale. First of all, we have no clear idea of how well the teachers could define each pupil's level of proportional reasoning in each problem type and subject. Secondly, there is no information concerning the teachers' ability to estimate the standard of problems which require proportional reasoning in other than mixture problems. Thirdly, we have no clear idea of how the strategies, inferred from the tests; if the teacher finds them out, can be utilized in the development of proportional reasoning. Fourthly, we have no clear idea how widely the strategies and procedural skills in the mixture problems can be generalized and transferred. Fifthly, there probably will be a need for developing the general attitudes of the mathematics teachers towards research on mathematics education and its possible applications.

There certainly exist still many more factors that will slow down and limit the actual application of the results of this project to school practice. When the differences and connections between the classroom tests and the interview connected with this study are solved a need particularly for instructional studies will arise. Furthermore, at the second stage of the project our purpose is to examine the influences of the task context on the performance processes of the pupils. At the third

stage of the project an attempt will be made to create a detailed mastery learning system for proportional reasoning and the idea of ratio as well as to test it in practice. Only then will it be possible to see, on research level, how and to what extent proportional reasoning can be developed in connection with certain problem types and topics during the lower and upper grades of the comprehensive school. At the same time we aim to study the possibilities to utilize the computer as an instrument of individual learning.

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Appendix I

Figure 1. The item combinations, performance levels and the strategies leading to incorrect result patterns in the classroom test JUICE I

	Item combination/ performance levels	Correct pattern Answers	Strategies leading to a correct pattern	Incorrect pattern	Strategies leading to an incorrect pattern
IA	1. 4:1 vs 1:4	A	1-dimensional "Juices" or "Waters"	E	"Quantities" or "Differences" e.g. 4+1=1+4=5
			or		e.g. 4-1=4-1=3
	2. 1:2 vs 2:1	B	"Within"-scr "strong vs light"	E	
				B	Misunderstanding
				A	or
				B	Erroneous observation
				B	or
				A	Carelessness
				A	or
					A Guess
IB	3. 1:2 vs 1:5	A	"Between"-scr Juices the same, Waters comparison	E	1-dimensional "Juices" 1:1
	4. 1:1 vs 1:2	A		E	
	Note! In addition to this 3:1 vs 4:1 would have been required, so that the whole pattern would not succeed by 1-dim. waters		In principle, can be solved also by using the 1-dim. waters scr	B	"Quantities"
				B	1+2=3 1+5=6
				A	Misunderstanding
				B	or
				B	Erroneous observation
				A	or
					Carelessness or a Guess

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Figure 1. (continued)

	Item combination/ performance levels	Correct pattern Answers	Strategies leading to a correct pattern	Incorrect pattern	Strategies leading to an incorrect pattern
IC	5. 3:1 vs 2:2	A	"Within"-str "strong/ light" vs "suitable" In principle, the 1-dim. "Waters" arr will also lead to a correct result pattern	A	
	6. 2:1 vs 1:1	B		A	1-dimensional "Juices"
	7. 2:1 vs 3:3	A		B	
	8. 2:2 vs 3:4	A		B	
	Note! In addition to this, 3:2 vs 1:1 would have been required, so that not even in theory it would be possible to get the whole pattern by the 1-dim. waters str			B	"Quantities"
				A B	
				A or B	
				A B	"Differences" without taking any notice of direction
				A B	
				Other combi-nations	Misunderstanding or Erroneous observation or Carelessness or a Guess
	9. 4:3 vs 1:2	A	"Within"-str "strong" vs "light"	A	"Quantities" or 1-dimensional
	10. 3:4 vs 6:5	B		B	"Juices"
	11. 2:1 vs 3:4	A		B	
	Note! If we want to make a difference between the "quantities' and "1-dim. waters" strategies, the items 2:5 vs 4:3 and 2:3 vs 4:1 will be needed			A	1-dimensional "Waters"
				A	
				E	
				E	"Differences" without taking any notice of the direction
				E	
				Other combi-nations	Misunderstanding or Erroneous observation or Carelessness or a Guess

Appendix I (continued)

Figure 1. (continued)

	Item combination/ performance levels	Correct pattern Answers	Strategies leading to a correct pattern	Incorrect pattern	Strategies leading to an incorrect pattern
IIA	12. 1:1 vs 3:3	E	"Between"-str 3:3 = 3 (1:1)	B B A	1-dimensional "Juices" or Quantities
	13. 2:2 vs 3:3	E	"Between" or "Within" differences 3-1=3-1=2	A A	1-dimensional "Waters"
	14. 4:4 vs 2:2	E	1-1=3-3=0	B	
			"Within"-str "suitable" vs "suitable"	Other combi-nations	Misunderstanding or Erroneous observation or Carelessness or a Guess
IIB	15. 1:2 vs 2:4	E	"Between"-str 2:4 = 2·(1:2)	B B A	1-dimensional "Juices"
	16. 1:3 vs 2:6	E	"Within"-str 2/1 and 4/2=2/1 or 2 parts of water to 1 part of juice in both of the mixtures	A A	1-dimensional "Waters"
	17. 4:2 vs 2:1	E		B A A A	Between or within difference 2-1=1<6-3=3, 3-1=2<6-2=4
				Other combi-nations	Misunderstanding or Erroneous observation or Carelessness or a Guess
III	18. 2:3 vs 1:2	A	"Between"-str 1:2=2:4 vs 2:3 "Within"-str 1,5/1 vs 2/1 0,67 vs 0,5	A B A	1-dimensional "Juices" or "Quantities"
	19. 2:3 vs 3:4	B		B A	1-dimensional "Waters"
	20. 4:3 vs 3:2	B		B E E E	Between or within difference 3-2=1=2-1
				Other combinations Misunderstanding etc.	

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Appendix 2

Table 1. The number of pupils on each grade in the Training School of Hämeenlinna, University of Tampere, the number of pupils taking part in the tests and the loss.

Grade	Number of pupils	The number of pupils taking part in the tests	Loss	%
1.	60	53	7	12
2.	60	53	7	12
3.	57	54	3	5
4.	56	52	4	7
5.	57	57	0	0
6.	41	37	4	10
	331	306	25	8

Appendix 3

Classroom tests Juice I, Juice II and Juice III
The items chosen to the analyses of the experimental part and their sequence (in parenthesis)
N.B.! The Juice I test consisted originally of 30 items.

Juice I test

1. 4:1 vs 1:4 (1)
2. 1:2 vs 2:1 (3)
3. 1:2 vs 1:5 (2)
4. 1:1 vs 1:2 (4)
5. 3:1 vs 2:2 (5)
6. 2:3 vs 1:1 (6)
7. 2:1 vs 3:3 (8)
8. 2:2 vs 3:4 (11)
9. 4:3 vs 1:2 (7)
10. 3:4 vs 6:5 (10)
11. 2:1 vs 3:4 (13)
12. 1:1 vs 3:3 (9)
13. 2:2 vs 3:3 (12)
14. 4:4 vs 2:2 (14)
15. 1:2 vs 2:4 (16)
16. 1:3 vs 2:6 (18)
17. 4:2 vs 2:1 (20)
18. 2:3 vs 1:2 (15)
19. 2:3 vs 3:4 (17)
20. 4:3 vs 3:2 (19)

Juice II test

1. 2:2 vs 3:? (1)
2. 1:1 vs 3:? (4)
3. 4:4 vs 2:? (6)
4. 1:2 vs 2:? (2)
5. 1:3 vs 2:? (3)
6. 4:2 vs 2:? (5)

Juice III test

1. 1:1 vs ?:? 6 1 (1)
2. 1:1 vs ?:? 8 1 (2)
3. 2:2 vs ?:? 8 1 (5)
4. 2:2 vs ?:? 6 1 (6)
5. 2:1 vs ?:? 6 1 (3)
6. 2:1 vs ?:? 9 1 (4)
7. 2:4 vs ?:? 121 (7)
8. 2:4 vs ?:? 3 1 (8)

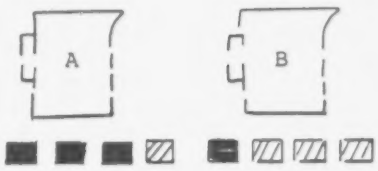
Appendix 4

The instruction connected with the classroom tests JUICE I, JUICE II and JUICE III with its model examples

Juice I test

3 glasses of juice and 1 glass of water will be poured into jug A
1 glass of juice and 3 glasses of water will be poured into jug B

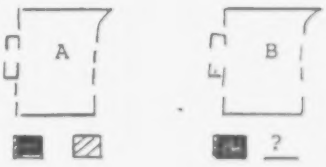
In which of the jugs will the mixture be stronger or will the mixture be equally strong?



Juice II test

1 glass of juice and 1 glass of water will be poured into jug A
1 glass of juice will be poured into jug B

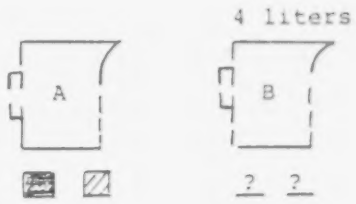
How many glasses of water should be poured into jug B, so that the mixtures would be equally strong (or would taste the same)?



Juice III test

1 liter juice and 1 liter water will be poured into jug A. We will get a 2-liter mixture.
We want to have 4 liters of an equally strong mixture in jug B

How many liters of juice and water should be poured into jug B?



Appendix 5

The form of the answer sheet in the classroom tests Juice I, Juice II and Juice III and the questions to be written on the blackboard

Juice I test

(1.)	A	B	E
(2.)	A	B	E
(3.)	A	B	E
	....		
(20.)	A	B	E

On the blackboard

In which of the jugs will the mixture be stronger, or will the mixture be equally strong in both jugs?

Ring A if in jug A
B if in jug B
E if in equally strong

Juice II test

(1.)	
(2.)	
	...
(6.)	

How many glasses of water should be poured into jug B, so that the mixture would be equally strong?

Juice III test

	J	W
(1.)		
(2.)		
(3.)		
	...	
(8.)		

How many liters of juice and water should be poured into jug B, so that the mixtures would be equally strong?

Appendix 6

Table 2. The criteria of the performance levels connected with the tests JUICE I, JUICE II and JUICE III

The ciriteria (A) of the JUICE I test: The subject will score one point in each of the item combinations IA-III, if all the answers to the problems of the combination are correct, otherwise 0 point.

The criteria (B) of the JUICE I test: Similar to the criteria (A) as regards the combinations IIB and III, one mistake permitted in the combinations IA-IIA; "humanity" -criteria.

The criteria (C) of the JUICE I test: One mistake permitted in the combinations IA, IB and IIA, two mistakes permitted in the combinations IC=IC & IC , in the combinations IIB and III as in the criteria (A) and (B).

The criteria of the performance levels in the JUICE II test:

I ("intuitive operations"), when the pupil does not solve correctly more than one of the items in the combination 1-3 (IIA)

IIA ("lower concrete operations"), when the pupil solves correctly at least two of the items in the combination 1-3 (IIA)

IIB ("transitional stage"), when the pupil has attained the preceding stage and solves correctly at least two items of the problems in the combination 4-6 (IIB)

IIB ("higher concrete operations"), when the pupil solves correctly all of the items 1-6

The criteria of the performance levels in the JUICE III test:

I ("intuitive operations"), when the pupil does not solve correctly more than two of the items in the combination 1-4 (IIA)

IIA ("lower concrete operations"), when the pupil solves correctly at least three of the items in the combination 1-4 (IIA)

IIB ("transitional stage"), when the pupil has attained the IIA level and solves correctly at least two of the items in the combination 5-8 (IIB)

IIB ("higher concrete operations"), when the pupil solves correctly all of the items 1-8.

RICHTLINIEN ZUM GEOMETRIEUNTERRICHT

Erkki Pehkonen

ZUSAMMENFASSUNG

Hier wird die Frage "Wie soll man Geometrie in der obligatorischen Schule (7-16 jährigen) unterrichten?" behandelt. Aus den entwicklungspsychologischen Forschungsresultaten werden einige Richtlinien für den Unterricht hergeleitet, die die Hauptprinzipien des Geometrieunterrichts genannt werden: Konkretisierung, Verbindung mit der Wirklichkeit, Entwicklung des räumlichen Vorstellungsvermögens, Schülertätigkeit, Unterrichten von Ganzheiten.

EINLEITUNG

Im Zusammenhang mit der Reform des Mathematikunterrichts (in den siebziger Jahren) wollte man die Geometrie aus der Machtstellung verdrägen, die sie in der Schulmathematik Hunderte von Jahren innegehabt hatte. Der Schwerpunkt des Mathematikunterrichts wurde zur Algebra und auf der gymnasialen Oberstufe zur Analysis hin verschoben. In etlichen Ländern hatte man sogar solchen Mathematikunterricht ausprobiert, in dem keine Euklidische Geometrie enthalten war.

Etwa zehn Jahre nach der Verwirklichung der Reform wurden aber die Folgen des Fehlens von Geometrieunterricht allmählich so deutlich, dass kaum jemand mehr an der Fehlerhaftigkeit der Reformmassnahmen zweifelte. Auf internationaler Ebene (u.a. auf dem Berkeley-Kongress 1980 und dem Mons-Kolloquium 1982) wurde deutlich festgestellt, dass der Geometrieunterricht angefangen von der Schule bis zur Lehrerausbildung wirkungsvoller Stützmassnahmen bedarf.

Über die Bedeutung der Geometrie ist in den letzten Jahren relativ viel geschrieben worden. In diesem Zusammenhang wird nicht auf die Frage eingegangen, warum Geometrie auch zum Unterricht der allgemeinbildenden Schule gehören sollte. Es gibt mehrere Artikel, in denen die Bedeutung der Geometrie hervorgehoben wurde - z.B. Bender (1983), Häggmark (1982), Pehkonen (1986) und Usiskin (1980).

Erich Wittmann legt in seinem Artikel (1974) die Richtlinien für den Geometrieunterricht dar. Er stützt sich auf die in der Carbondale-Konferenz für den Geometrieunterricht hergeleiteten Randbedingungen:

1° Der Geometrieunterricht soll die Bedeutung der Geometrie für das Alltagsleben und die anderen Teilgebiete der Mathematik hervorheben.

2° Der Geometrieunterricht soll die entwicklungspsychologischen Gesetzmässigkeiten berücksichtigen: der Geometrieunterricht auf der Primarstufe soll konkretes Arbeiten in konkreten Situationen sein.

Dazu ist zu bemerken, dass man Geometrie nicht als Fertigprodukt anbieten sollte; u.a. ist die deduktive Methode allmählich in Gebrauch zu nehmen (Wittmann 1974, 5).

In dieser Arbeit konzentriere ich mich innerhalb der obligatorischen Schule (7-16 jährige) auf die Frage "Wie sollte man Geometrie unterrichten?", wenn man von den erwähnten Wittmannschen Randbedingungen ausgeht.

Beim Konzipieren des Geometrieunterrichts (auch allgemeiner des Mathematikunterrichts) sollte man die Möglichkeiten des Schülers, den Unterricht entgegenzunehmen, vor Augenhaben. In erster Linie sind hier die entwicklungspsychologischen Ergebnisse über die Phasen der geistigen Entwicklung des Kindes zu berücksichtigen. Die Lehrinhalte und -methoden sollten so ge-

wählt werden, dass sie den bekannten Phasen der geistigen Entwicklung des Kindes optimal entsprechen.

Im folgenden werden einige Richtlinien zum Geometrieunterricht vorgestellt, die man aus den psychologischen (und fachdidaktischen) Forschungsresultaten ableiten kann. Die Unterrichtsprinzipien sollte man sowohl beim Entwickeln der Lehrpläne (beim Wählen der Lerninhalte) als auch beim Erörtern der Unterrichtsmethoden und - materialien berücksichtigen.

Die dargestellten Prinzipien werden sich einander teilweise überlappen, aber ich wollte sie in dieser Form einführen, damit die aus den verschiedenen Forschungsergebnissen hergeleiteten Bedingungen betont zur Geltung kommen.

KONKRETISIERUNG

Innerhalb der Schulmathematik kann man von abstrakter und konkreter Mathematik sprechen. In der Sekundarstufe I gehört z.B. zur abstrakten Mathematik u.a. die Buchstabenrechnung (Polynome, usw.); die konkrete Mathematik ist dagegen auf das Alltagsleben ausgerichtet.

In dem Parallelschulsystem lag im Gymnasium das Hauptgewicht wohl auf der abstrakten, formalen Mathematik, wogegen man sich in den Mittel- und Volksschulen (in der Bundesrepublik in den Real- und Hauptschulen) wohl mehr auf konkrete Mathematik konzentrierte. Weil in der finnischen obligatorischen Schule seit Herbst 1985 die ganze heterogene Altersklasse (nach der Auflösung des Niveaukurssystems) zusammen nach denselben Lehrplänen unterrichtet wird, müsste man sich auch in der Schulpraxis an diese Teilung erinnern. (Mehr über das finnische Schulsystem findet man z.B. in Pehkonen, 1983).

Ohne Zweifel kann man die Geometrie (in der durch Lehrplan bestimmten Form) hauptsächlich zur konkreten Mathematik rechnen. Der Geometrieunterricht kann auch sehr leicht zu abstrakt werden, weil der Lehrer seine eigenen Erfahrungen (aus dem Gymnasium) über das Geometrielernen mitbringt. Darum ist die Bedeutung des Lehrbuches als Führer zur Konkretisierung besonders wichtig. Besonders die Geometrie ist ein solches Teilgebiet der Mathematik, das man leicht auf einem anschaulichen Grund aufbauen kann.

Galperinsche Lerntheorie

Mit konkretem Unterricht meine ich (auch auf der Sekundarstufe I) das Anleiten des Schülers zum konkreten Arbeiten: Der Schüler handhabt Gegenstände, baut Modelle, führt Messungen durch usw. Erst wenn dieses konkrete Arbeiten gründlich eingeübt wurde, kann man zu abstrakteren Arbeiten übergehen: Figuren auf Papier zeichnen usw.

Dieser Gedanke stützt sich teilweise auf die Galperinsche Lerntheorie. Nach Pjotr J. Galperin geschieht das Lernen geistiger Handlungen in fünf Etappen (1969): Orientierungsphase, materielle Phase, sprachliche Phase, Phase der inneren Sprache, Phase der Verinnerlichung. Oft werden sie zu drei Etappen vereinfacht:

(1) Konkrete Phase,
(2) Phase der äusseren Sprache,
(3) Phase der inneren Sprache.

Die erste Phase bedeutet - dem Namen nach - konkretes Arbeiten, durch das man mit der Sache vertraut wird. In der nächsten Phase wird die Handlung noch nicht völlig beherrscht, wobei der Schüler sich erklären (= sprechen) kann, was er zu tun hat. (Wie ein jeder weiss, hilft lautes Sprechen dabei, sich auf eine fremde Sache zu konzentrieren.) In der dritten Phase ist die

Handlung schon automatisch geworden; an die Stelle der äusseren Sprache tritt die innere (= das Denken).

Vom Standpunkt der Schule aus ist es wesentlich, dass die Schüler beim Lernen keine Phase überspringen können. Falls man im Unterricht zu schnell (vom Standpunkt des Schülers aus) die konkrete Phase verlässt, so hat dies im allgemeinen (für den Schüler) Lernschwierigkeiten zur Folge. Dagegen kann ein Erwachsener direkt mit einer höheren Phase einsetzen, weil seine breitere Erfahrungsbasis die benötigte Stütze gibt.

Entwicklungsstufen nach Piaget

Zuerst rufen wir uns die von Jean Piaget eingeführten Stufen der geistigen Entwicklung von Kindern in Erinnerung (siehe z.B. Wittmann 1981, 74-77): Auf der Stufe der konkreten Operationen gründet sich das Denken auf solche Begriffe, die in unmittelbarer Verbindung mit dem Konkreten stehen. Der Hauptunterschied zwischen den Stufen der formalen und konkreten Operationen liegt darin, dass auf ersterer auch solche Sachverhalte behandelt werden können, die möglich, aber nicht unbedingt real sind.

Beim Beschreiben der Grenzen des Fortschritts auf der Stufe der konkreten Operationen äussert David P. Ausubel (1963, 117):

> "Der verständnisvolle Umgang mit Beziehungen zwischen Abstraktionen oder mit Begriffen von Begriffen ist aber etwas ganz anderes. Dabei muss sich das Grundschulkind auf unmittelbare oder kurz vorher erfolgte konkrete Erfahrungen stützen können: wenn solche Erfahrungen nicht vorliegen, kann das Kind abstrakte Aussagen über Beziehungen nicht auf seine kognitive Struktur beziehen und steht ihnen daher verständnislos gegenüber."

Der wichtigste Unterschied zwischen den Stufen der konkreten Operationen und der formalen Operationen liegt (wie früher schon erwähnt wurde) in der Möglichkeit, solche Sachen behandeln zu können, die keine realen Entsprechungen haben. Besonders anfangs besteht auf der Stufe der formalen Operationen die Gefahr, dass sich der Schüler beim ausschliesslichen Operieren auf der abstrakten Ebene nicht immer die Begriffe aneignen kann. Ausubel charakterisiert die Stufe der formalen Operationen folgendermassen (ibid. 118):

> "Mit Beginn dieser Stufe werden die Kinder zunehmend unabhängiger von Erfahrungen mit konkretem Material, wenn sie komplexe abstrakte Beziehungen verständnisvoll in ihre kognitive Struktur einbeziehen sollen. Schliesslich ... taucht eine qualitativ neue Fähigkeit auf: das intellektuell reife Individuum wird fähig, abstrakte Beziehungen ohne jeden Bezug zur konkret erfahrenen Wirklichkeit zu verstehen."

Im allgemeinen wird behauptet, dass sich derr Übergang von der Stufe der konkreten Operationen zur Stufe der formalen Operationen etwa im Alter von 12 Jahren vollzieht; es gibt einige, bei denen dieser Übergang sehr spät oder sogar niemals eintritt (siehe z.B. Wittmann 1981, 70). Aber nach einigen in Grossbritannien und Australien durchgeführten Untersuchungen scheint die Entwicklung langsamer zu verlaufen. L.G. Dale stellt in seinem Artikel (1975, 122) ein Schema (Abb. 1) dar, das nach ihm ein realistischeres Bild von der Situation gibt. Die sich schnell entwickelnden Kinder haben schon im Alter von 12 Jahren die Stufe der formalen Operationen erreicht, während die sich durchschnittlich entwickelnden Kinder sie erst im Alter von 17 Jahren erreichen und die sich langsam entwickelnden vielleicht niemals.

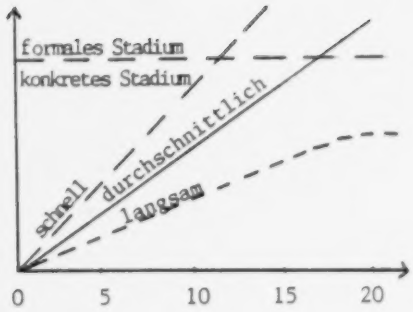


ABBILDUNG 1.

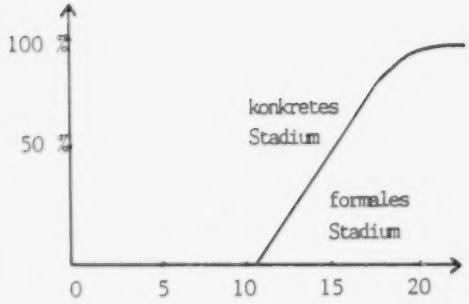


ABBILDUNG 2.

Danach wird die Situation vom Standpunkt der ganzen Altersklasse aus betrachtet: In Abbildung 2 legt Dale dar, wie gross der Anteil der Kinder in jedem Lebensjahre ist, die die Stufe der formalen Operationen erreicht haben (ibid. 123). Nach diesem Schema haben von den 12-jährigen etwa 15 % und von den 15-jährigen weniger als Hälfte die Stufe der formalen Operationen erreicht.

Die Piagetsche Theorie ist von mehreren Forschern sowohl aus dem Westen als auch aus dem Osten vehement kritisiert worden (siehe z.B. Resnick et al. 1981, 174-186). Obgleich die Theorie nicht unerschütterlich ist, gibt sie in jedem Fall wertvolle Informationen für die Unterrichtspraxis: Auch auf der Sekundarstufe I sind nicht alle Schüler unbedingt bereit, abstraktes Denken durchzuführen. Also ist die Konkretisierung beim Unterrichten der ganzen Altersklasse (in den heterogenen Klassen der Gesamtschule) eine notwendige Bedingung für einen erfolgreichen Unterricht.

Prinzip der Konkretisierung

Nach den Piagetschen Untersuchungen befinden sich fast alle Schüler der Primarstufe auf der Stufe der konkreten Operationen; sie brauchen also einen konkreten Ausgangspunkt für ihr Denken. Die erwähnten Forschungsresultate von DALE zeigen, dass auch der grösste Teil der Schüler auf der Sekundarstufe I immer noch auf der Stufe der konkreten Operationen ist.

Hieraus folgt, dass auf der Primarstufe
(1) der Geometrieunterricht konkretes Arbeiten sein soll.
Noch auf der Sekundarstufe I muss der Lehrer reichlich Zeit für die Einführung einer neuen Sache (eines neuen Begriffs) veranschlagen und besondere Aufmerksamkeit der Konkretisierung und Veranschaulichung des Unterrichts schenken. (In dem Parallelschulsystem sind die lernschwächsten Schüler, die einen konkreten Ausgangspunkt für ihr Denken benötigen, nach der Primarstufe gewöhnlicherweise zur Hauptschule übergewechselt, wo das Prinzip der Konkretisierung unbedingt verwirklicht werden sollte.

Das Prinzip der Konkretisierung ist eigentlich das Streben, vom Konkreten zum Abstrakten überzugehen. Dies geschieht am besten, wenn man dem Handeln auf der konkreten Ebene genügend Möglichkeiten (vom Standpunkt des Schülers aus) einräumt, bevor man zur abstrakten Ebene überzugehen versucht. Also könnte man das Prinzip auch als "Begleiten der geistigen Entwicklungsphase" bezeichnen.

Leider ist die Berücksichtigung der Entwicklungsstufe des Schülers beim Wählen der Lerninhalte und -methoden oft nur zweitrangig. Hieraus folgt sehr wahrscheinlich unnötiger kognitiver Ballast und Störungen im affektiven Bereich (Wittmann 1981, 82). Sinnbildlich gesagt, dem Schüler werden beim Puzzle-Spielen solche Teilstücke gegeben, die er nicht einmal gebrauchen kann. Deshalb sollte die Konkretisierung ein den ganzen Unterricht der Pflichtschule konsequent durchdringendes Prinzip sein.

VERBILDUNG MIT DER WIRKLICHKEIT

In den letzten Jahren ist das Prinzip des praxisnahen Unterrichts betont worden; im Anschluss daran ist auch reichlich Material publiziert worden. Heinrich Winter hat Begründungen für die Forderung der "Umwelterschliessung im Geometrieunterricht" gesammelt und sie in der Einführung der von ihm herausgegeben Zeitschrift "Der Mathematikunterricht" veröffentlicht (1978, 5-6):

- Die Erfahrungsrealität könnte der wichtigste Ausgangspunkt für geometrische Fragen sein.
- Die Realitätsnähe wirkt stärker auf die Intuition des Schülers als die Anwendung der Geometrie innerhalb der Mathematik.
- Die Schüler können die praktische Bedeutung der Geometrie kennenlernen.
- Die Realitätsnähe stärkt die Konzentration des Denkens und verhindert, dass der Geometrieunterricht zum blossen Terminologiekurs verkommt.
- Die Realitätsnähe zwingt ständig zum neuen Werten und Ordnen der Kenntnisse.

Dieses Verbinden mit der umgebenden Wirklichkeit im Geometrieunterricht stellt auch Jürgen Jayen heraus. Er betont, dass mit Hilfe der Geometrie besonders den langsam Mathematik lernenden Schülern die Möglichkeit gegeben werden kann, die Herausforderungen ihrer Umgebung ind ihres Berufs anzunehmen. Hayen benutzt in diesem Zusammenhang die Bezeichnung "Wirklichkeitsgeometrie". Seiner Mainung nach kann man solchen Geometrieunterricht in jeder Klassenstufe üben (Hayen 1982, 31-32).

Die Kopplung mit der Alltagswirklichkeit haben besonders Bender (1978) und Häggmark (1982) betont. Die neue deutsche Zeitschrift "Mathematiklehren" beleuchtet diesen Aspekt durch Themanummern, die sich auf die mathematische (und mathematik-didaktische) Behandlung bestimmter Alltagsphänomene (wie Spiegel, Sport, Fliegen) konzentrieren. Am stärksten ist die Verbindung zwischen

der Geometrie und dem Alltagsleben in den von dem niederländischen IOWO-Institut publizierten Geometriebüchern herausgestellt worden; viele Ideen in dieser Hinsicht findet man u.a. in den Artikeln Freudenthal (1981), Schoemaker et al. (1981) und de Lange (1982).

Über Möglichkeiten in der Praxis

Für jedes Individum ist es wichtig, den realen Raum, in dem man sich bewegt, möglichst gut kennenzulernen. Darum sollte der Geometrieunterricht (und auch allgemeiner der Mathematikunterricht) in jeder Klasse stark an die Wirklichkeit gekoppelt sein.

Schaut man zum Fenster heraus, sieht man in der Natur und in der künstlichen Umgebung Geometrie in Hülle und Fülle, wenn man nur die Augen offenhält. Man könnte sogar die Schüler zu "Formenjägern" ausbilden; wenn im Unterricht eine neue Figur oder ein neuer Körper durchgenommen wird, könnte die erste Hausaufgabe folgendermassen lauten: Wo hast Du die Figur (oder den Körper) X auf deinem Heimweg schon einmal gesehen?

Andererseits kann man die Wirklichkeit, z.B. in der Form von Lebensmittelverpackungen, auch in die Klasse hereinbringen. Es wäre nützlich, in der Klasse eine reichhaltige Kollektion von Lebensmittelverpackungen zu sammeln. Ein Kaffeepaket ist für den Schüler ein besseres Modell eines Quaders als das entsprechende nüchterne Schulmodell. Desgleichen ist die Milchtüte ein den Schülern vertrauteres Litermass als das Messglas der Physikklasse. (Man muss nur vorher überprüfen, ob die Milchtüte tatsächlich genau einen Liter enthält.)

Es ist nützlich, auch andere Gegenstände unseres alltäglichen Lebens im Unterricht heranzuziehen. Z.B. kann man von einer einfachen Briefwaage zur Viereck Geometrie übergehen, weil die Funktion der Briefwaage auf beweglichen Vierecken basiert (Abb. 3). Es liesse sich sogar denken, dass man die Behandlung

der Vierecke aus der Untersuchung der Briefwaage (oder ähnlicher Gegenstände unseres alltäglichen Lebens) ableiten könnte: Zuerst könnte man ein prinzipielles Pappmodell der Briefwaage anfertigen, dessen Variationen ausprobieren, und danach könnte man zu den Vierecken übergehen usw.

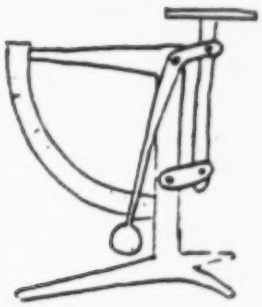


ABBILDUNG 3.

Prinzip der Verbindung mit der Wirklichkeit

Also sollte man in der Pflichtschule

(2) den Geometrieunterricht immer mit den Alltagssituationen koppeln.

Dieses Prinzip - Umwelterschliessung im Geometrieunterricht - ist gerade der Ausgangspunkt der Galperinschen Lerntheorie (S. 38). In jeder Klasse kann (und soll) man im Geometrieunterricht die Verbindung mit der umgebenden Wirklichkeit pflegen (Bender 1978, 26). Die Betrachtung der Umgebung unter dem Aspekt der Geometrie bedeutet, die Struktur des physikalischen Raumes zu erforschen, und das Hauptziel ist, die Anwendbarkeit und Möglichkeiten dieser Struktur zu verstehen.

ENTWICKLUNG DES RÄUMLICHEN VORSTELLUNGSVERMÖGENS

Mit der Abnahme des Geometrieunterrichts (im Zusammenhang mit der Reform des Mathematikunterrichts) hat man neben den Schwierigkeiten in den Grundbegriffen auch ein Abschwächen der Visualisierungsfähigkeit beobachtet (Bishop 1981, 166). Die Schule scheint die Schüler schlecht darauf vorzubereiten, der normalen dreidimensionalen Welt zu begegnen.

Bekanntlich ist das räumliche Vorstellungsvermögen ein Faktor der Intelligenz. Dieser Hauptfaktor lässt sich in drei Unterfaktoren zerlegen: Veranschaulichung, Räumliche Beziehungen und Räumliche Orientierung (Becker 1980, 47).

Bedeutung des Visualisierens

In der Mathematik braucht man nach Alan Bishop zwei verschiedene, einander ergänzende Denkweisen: kreatives Denken, für das "Intuition" typisch ist, und analytisches Denken, für das "Logik" typisch ist. Diese Denkweisen kann man foldendermassen charakterisieren: Zur Logik gehört Verbalisieren und zur Intuition Visualisieren. Das Arbeiten mittels Wörtern (verbal) ist immer eindimensional, während das Denken mittels Bildern (visuell) im allgemeinen zwei- oder dreidimensional ist (Bishop 1981, 167-168).

Bei seiner Erörterung der durch den Geometrieunterricht ermittelten Fertigkeiten hat Alan Hoffer auch die visuellen Fertigkeiten angesprochen (1981, 11). In seinem Artikel verweist Hoffer auch auf die Untersuchungen, die die Bedeutung der verschiedenen Gehirnhälften für das Mathematiklernen zum Thema gehabt haben (u.a. Wheatly et al. 1978). In der Tabelle (Abb. 4) ist die Aufgabenteilung der Grosshirnhemisphären dargestellt worden (Kane 1979). Dort sieht man, dass der grösste Teil der geometrischen Handlungen in der rechten Gehirnhälfte lokalisiert

ist; besonders ist der anteil der Visualisierung und des räumlichen Wahrnehmens zu bemerken.

Linke Gehirnhälfte	Rechte Gehirnhälfte
Rechtes Sichtfeld Bewegungen des rechten Auges	Linkes Sichtfeld Bewegungen des linken Auges
	Anpassung an den räumlich-geometrischen Raum Finden von Routen in Räumen
Benennen von Räumen und Objekten	Lokalisieren verschiedener Objekte auf der Ebene Tiefenvisuelle Wahrnehmung
	Gesamtgestalten
Zusammensetzung des Raumes aus Teilen	Gleichzeitiges Vorstellen mehrerer visueller Gestalten
	Erkennen geometrischer Formen
Wahrnehmung von Einzelheiten	Zeichnen
	Ein Modell nach Schema oder Bild bauen
Benennen von Gesichtern	Erkennen von Gesichtern

ABBILDUNG 4.

Die meisten Kleinkinder sind nach der rechten Gehirnhälfte hin orientiert. Wenn die Kinder in die Schule eintreten, haben sie ein gutes intuitives Sinnbild vom Raum. Die heutigen Mathematikkurse tun aber sehr wenig, um die raumbegriffe zu entwickeln

Heinrich Besuden meinte, in der Schule genüge es nicht, das räumliche Vorstellungsvermögen zu berücksichtigen, sondern man müsse und könnte es entwickeln und fördern (1974, 49). Die Entwicklungsmöglichkeiten der Raumvorstellung sind besonders im Alter der Primarstufenschüler optimal, man hat aber mit systematischem Üben auch bei älteren Schülern (in der 8. Klasse) gute Resultate erreicht (Barke 1983, 353).

Prinzip der Entwicklung des räumlichen Vorstellungsvermögens

Weil das räumliche Vorstellungsvermögen sich sehr langsam entwickelt und viele Erfahrungen verlangt, sollte
(3) die Entwicklung des räumlichen Vorstellungsvermögens
ein den gesamten Geometrieunterricht der Pflichtschule durchdringendes Prinzip sein. Auf keiner Stufe darf die Geometrie sich nur auf Zweidimensionalität beschränken! Den selben Gedanken kann man auch bei Heinrich Wölpert lesen (1983, 9).

SCHÜLERTÄTIGKEIT

Das Ziel des Mathematikunterrichts ist nicht allein die sichere Rechenkunst, sondern auch die vielseitige Entwicklung des mathematischen Denkens. Darum hat man allgemein das Prinzip des aktiven Lernens akzeptiert (Wittmann 1981, 77), nach dem der Unterricht erfolglos bleibt, wenn der Stoff nicht von den Schülern aktiv bearbeitet wird. Ein natürlicher Weg zu den aktiven Lernprozessen ist, das Problemlösen in den Mathematikunterricht mit einzubeziehen. Die Idee, die eigenständige Aktivität des Kindes zu benutzen, ist auch in den Schriften von Jean Piaget zu finden (z.B. 1972, 139-141).

Auch beim operativen Prinzip kann man die Betonung der Schüleraktivität beobachten; dies hat Erich Wittmann besonders her vorgehoben:

(del Grande 1983, 164). Gerade der Geometriekurs hat aber die Möglichkeit (und er sollte sie auch nutzen), die Schüler mit genügend Situationen zu konfrontieren, die beide Gehirnhälften entwickeln helfen.

Raumvorstellung

Das räumlich Vorstellungsvermögen entwickelt sich allmählich: Bei den 9-10 -jährigen macht es etwa die Hälfte und bei den 12-14 -jährigen etwa 80 % von dem durchschnittlichen Leistungsvermögen eines Erwachsenen aus (Rost 1977, 47). Im allgemeinen ist man der Ansicht, dass sowohl die Herkunft als auch die Umgebung auf die Entwicklung der Intelligenz einwirken; noch ist es ungeklärt, wie hoch der anteil von beiden ist. Nach der Untersuchung von Hans-Dieter Barke besitzt der grösste Teil der Schüler erst in der 8. Klasse ein so weit entwickeltes Raumvorstellungsvermögen, dass es für den naturwissenschaftlichen Unterricht ausreicht (1980, 129).

In seiner Untersuchung über das Fördern der Raumvorstellung im Chemieunterricht hat Barke Gedanken über die Bedeutung des räumlichen Vorstellungsvermögens zusammengestellt (1983, 352-353; hier auch die Verweise auf die Originalquellen):

- In der Industrie haben die Raumvorstellungstests grössere Korrelation mit dem Erfolg im Beruf als die Intelligenztests.
- Mehrere Untersuchungen zeigen, dass die Raumvorstellung bedeutungsvoll für den Schulerfolg ist.

Es gibt Untersuchungen (u.a. Tobias 1978), die die Beziehung zwischen einem schwachen Raumvorstellungsvermögen und der Mathematikangst belegen. Es zeigt sich, dass die Schüler, die Angst vor dem Mathematiklernen haben, auch die die Raumvorstellung verlangenden Aufgaben nicht gut ausführen können.

> "Medien, die nicht "bearbeitet", sondern nur betrachtet werden können, haben daher nur sehr beschränkten Wert" (Wittmann 1981, 79).

Derselbe Gedanke kommt auch im Motto des englischen Nuffield-Projekts vor: I hear, and I forget - I see, and I remember - I do, and I understand.

Jeder Mensch erhält den grössten Teil seines Wissens an erster Stelle durch die Augen und an zweiter Stelle durch die Ohren; der anteil der übrigen Organe ist verschwindend klein. Nach den Forschungsresultaten der amerikanischen audiovisuellen Gesellschaft bleibt einem im Gedächtnis:

10 % von dem, was man liest
20 % von dem, was man hört
30 % von dem, was man sieht
70 % von dem, was man hört und sieht
80 % von dem, was man selber sagt
90 % von dem, was man selbst ausführt (Pundt et al. 1979, 40).

Prinzip der Schülertätigkeit

Also sollte man im Unterricht (und nicht nur im Geometrieunterricht) (4) die eigenständige Tätigkeit der Schüler betonen. Die Aktionsformen sollten variert werden, damit das Interesse des Schülers wachgehalten wird. Die Schüleraktivitäten sollte man problemzentrisch aufbauen, damit die Entwicklung des mathematischen Denkens bei den Schülern ermöglicht und gefördert wird. Die aktivitäten sollten nicht Selbstzweck sein.

UNTERRICHTEN VON GANZHEITEN

Vom Standpunkt des Lernens ist es effektiver, den zu unterrichtenden Stoff zu Ganzheiten zusammenzufassen, als ihn in getrennten Teilen zu behandeln. Solche Ganzheiten können vom Standpunkt des Schülers auch sehr anspruchvolles Wissen enthalten, d.h. dessen Aneignung kann vom Schüler eigenständige Bemühungen verlangen. Dieser Gedanke wird auch von der van Hieleschen Entwicklungsniveautheorie gestützt.

Über die Niveaus der geistigen Entwicklung in der Geometrie

Im Jahre 1959 publizierte Pierre M. van Hiele die Resultate seiner Untersuchungen, die die geistige Entwicklung beim Lernen der Geometrie behandelten. Dabei stellte er die folgenden fünf Entwicklungsniveaus vor (van Hiele 1959; siehe auch Hoffer 1981 und Silfverberg 1985):

Niveau I : Erkennen,
Niveau II : Analysieren,
Niveau III: Ordnen,
Niveau IV : Schlussfolgern,
Niveau V : Verstehen des Axiomsystems.

Auf dem ersten Niveau (Erkennen) lernt der Schüler Benennungen und erkennt die Gestalt als ein Ganzes. Z.B. erkennt der Schüler ein Rechteck unter anderen Vielecken, aber er ist sich vielleicht vieler Eigenschaften des Rechtecks nicht bewusst.

Auf dem zweiten Niveau (Analysieren) analysiert der Schüler die Eigenschaften der Figuren. Z.B. weiss der Schüler, dass die entgegengesetzten Seiten eines Rechtecks gleich lang sind, aber er weiss nicht, wie Rechtecke sich zu Quadraten oder rechteckigen Dreiecken verhalten.

Auf dem dritten Niveau (Ordnen) ordnet der Schüler Figuren logisch zu und versteht die Verbindungen zwischen Figuren und die Bedeutung exakter Definitionen. Z.B. versteht der Schüler, warum jedes Quadrat auch ein Rechteck ist, aber er kann vielleicht nicht erklären, warum die Diagonalen des Rechtecks gleich lang sind.

Auf dem vierten Niveau (Schlussfolgern) versteht der Schüler die Bedeutung der Schlussfolgerung sowie die Bedeutung der Sätze und des Beweises. Z.B. kann der Schüler den Kongruenzsatz "SWS" benutzen, um Behauptungen über Rechtecke zu beweisen, aber er versteht nicht, warum es notwendig ist, den vorliegenden Kongruenzsatz zu begründen.

Auf dem fünften Niveau (Verstehen des Axiomsystems) versteht der Schüler die Bedeutung der Exaktheit beim Behandeln von Grundlagen der Geometrie sowie die Beziehungsrelationen swischen Strukturen. Z.B. versteht der Schüler, wie das Parallelaxiom sich zu der Existenz der Rechtecke verhält und dass es innerhalb einer nicht-euklidischen Geometrie keine Rechtecke gibt. Dieses anspruchvollste Niveau wird in der Schule selten erreicht.

Fortsetzung der van Hieleschen Arbeit

Die Sowjets interessierten sich sofort für die van Hielesche Theorie. Nach intensiver Forschungsarbeit in den Jahren 1960-64 stellte die Akademie der pädagogischen Wissenschaften der Sowjetunion fest, dass die von van Hiele aufgestellten Behauptungen und Prinzipien stichhaltig sind (del Grande 1983, 163). Die sowjetischen Forschungen zeigten auch, dass alle Schüler durch richtig geordnetes Material schon gegen Ende des 3. Schuljahres Niveau II erreichen können (Pyshkalo 1968).

Die van Hieleschen und sowjetschen Forschungen haben gezeigt, dass der Übergang von einem Niveau zum nächsten nicht von der Reife oder dem Alter der Schüler abhängt. Manche Erwachsene sind

noch auf Niveau I, weil sie nicht genügend Erfahrungen gesammelt haben, um Niveau II zu erreichen. Der Lehrer kann den Übergang von einem Niveau zum nächsten durch die richtige Wahl des Lehrstoffs und der Unterrichtsmethode beschleunigen (del Grande 1983, 164).

Die van Hieleschen Niveaus bilden eine hierarchische Struktur, über die Freudenthal festgestellt hat, dass "die Handlung jedes Niveaus das Objekt der Analyse auf dem nächsten Niveau ist" (Freudenthal 1973). Diese Struktur lässt sich folgendermassen beschreiben: Die Objekte der Analyse sind

- auf Niveau I die geometrischen Grundelemente,
- auf Niveau II die eigenschaften der Grundelemente,
- auf Niveau III die Stäze, die die Beziehungen zwischen den Eigenschaften beschreiben,
- auf Niveau IV die Folgen dieser Sätze,
- auf Niveau V die Eigenschaften dieser Folgen.

Harry Silfverberg hat neulich untersucht, wie die finnischen Schüler im heutigen Geometrieunterricht die van Hieleschen Niveaus erreichen. Getestet hat er zwei fünfte Klassen (N = 44), zwei siebente Klassen (N = 38) sowie eine achte (N = 21) und eine neunte Klasse (N = 18). Im folgenden wird ein Teil seiner Resultate tabellarisch wiedergegeben (Silfverberg 1984, 27-28, 54; siehe auch Silfverberg 1985):

Bezüglich des Dreieck- und Viereckbegriffs sind von den Versuchspersonen

	5.Kl	7.Kl	8.KL	9.KL
mindestens auf Niveau II	20 %	35 %	50 %	60 %
mindestens auf Niveau III	2 %	10 %	20 %	10 %

In der Beherrschung der Begriffe Dreieck, Quadrat, Rechteck und Parallelogramm erreichen von den Versuchspersonen

	7.KL	8.KL	9.KL
mindestens Niveau II	15 %	35 %	35 %
mindestens Niveau III	3 %	10 %	-

Auf Grund dieser Untersuchung scheint der finnische Geometrieunterricht nicht besonders effektiv zu sein.

Über Lernstrategien

Nach der neuesten Lernforschung wird Lernen als Informationsverarbeitung betrachtet, die aus vielen Handlungen besteht: aus dem Wahrnehmen, dem Richten der Aufmerksamkeit, dem Gliedern der Wahrnehmungen, dem Kombinieren der neuen Information mit den früheren Kenntnissen, aus dem so geschaffenen neuen Wissen und aus dem Wiedergeben dieses Wissens entweder mündlich, schriftlich oder in anderer Weise. Als Folge einer solchen Informationsverarbeitung tritt als Lernprodukt also eine Änderung darin ein, wie der Mensch die Wirklichkeit oder deren Erscheinungen gliedert und wie er die verschiedenen Begriffe versteht.

In den letzten Jahren hat man sein Augenmerk darauf gerichtet, wie sich das Wissen im Kopf der Schüler gliedert. Es zeigte sich, dass die Wissensstrukturen bei den langsam und den schnell Mathematik lernenden Schülern qualitativ verschieden sind.

An der Universität Turku sind die Wissensstrukturen von Schülern verschiedener Leistungsniveaus ermittelt worden. Einige Schüler können ihr Wissen in logische hierarchische Strukturen gliedern. Dagegen nehmen andere ihr Wissen als völlig zufällige Assozia-

tionen wahr. In Leistungssituationen brechen die letzteren Wissenstrukturen unter dem Evaluationsdruck zusammen (Lehtinen 1982, 80-82).

Vom Standpunkt des Lernens aus ist es also wesentlich, dass der Lehrer beim Unterrichten einer Stoffganzheit bewusst versucht, die Wissensstruktur des Schülers hierarchisch auf zu bauen. Falls die Wissensstrukturen einiger Schüler auf der Assoziationebene bleiben, sollte der Lehrer versuchen, in seinem Unterricht bewusst die Ganzheit zusammenzusetzen.

Prinzip des Unterrichtens von Ganzheiten

Nach van Hiele sollte man auf jedem Entwicklungsniveau genügend lange Zeit arbeiten, damit dem Schüler die Möglichkeit gegeben wird, seine Kenntnisse neu zu ordnen. In der Tat ist das Lernen von Einzelheiten kein Lernen in der tieferen Bedeutung des Wortes, sondern das wirkliche Lernen vollzieht sich erst im Hirn des Schülers, wenn er neue Informationen zu Ganzheiten ordnet (Wittmann 1981, 81). Hieraus wird das folgende Prinzip abgeleitet: In der Geometrie (wie auch allgemein in der Mathematik)
(5) sollen Ganzheiten unterrichtet werden.
Die Teile der Geometrie dürfen nicht unter die übrige Mathematik aufgesplittert sein; bei jedem Teilgebiet muss man deutlich sagen können, in welchem Schuljahr dessen Begriffe erlernt werden.

FAZIT

In den obigen Kapiteln sind die aus den entwicklungspsychologischen Forschungsresultaten hergeleiteten Hauptprinzipien zum Geometrieunterricht dargestellt worden. Hieraus kann man einige Folgerungen zur Schulgeometrie ableiten, sog. Randbedingungen für den Geometrieunterricht in der Schule. In einer früheren Arbeit (Pehkonen 1985) hat der Verfasser versucht, die Verwirk-

lichung dieser Prinzipien im Geometrieunterricht der obligatorischen Schule zu skizzieren.

In der fachdidaktischen Literatur findet man viele Modelle, wie der Unterricht nach den erwähnten Lehrprinzipien verwirklichtwerden könnte: Z.B. wenn die Schüler sich selbständig mit Lebensmittelverpackungen beschäftigen, werden die vier ersten Hauptprinzipien beachtet. Andererseits kann man bei solchen Beschäftigungen im Unterricht relativ leicht versuchen, sowohl die Rechenkunst als auch das mathematische Denken zu entwickeln und die Kreativität zu fördern.

International gesehen ist die Situation in der Schulgeometrie überall ziemlich gleich: Weil die Position der Geometrie nicht gefestigt ist, betonen die Lehrer in ihrem Unterricht andere Teilgebiete der Mathematik und reduzieren den Geometrieunterricht. Diese Situation drückt Jean Pedersen treffend aus, der in den Vereinigten Staaten und auf seinen Reisen in Südostasien und Europa viele Gespräche mit Lehrern geführt und hieraus ein Gesamtbild von der Lage des geometrischen Schulunterrichts zusammengefügt hat: "... nach meiner Einsicht ist die Lage der Geometrie sehr gleichwertig in allen Ländern. Kurz gefasst: Geometrie ist unterbeschäftigt!" (Pedersen 1983, 158).

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COMPUTERS AND GEOMETRY TEACHING

Günter Graumann

ABSTRACT

At the beginning of my article I wish to emphasize the relevance of geometry learning in education and present my conception of geometry teaching. Thereby I shall point out why I don't find the orientation not only in Euklid but also in modern axiomatism sensible. A conception of geometry teaching which suits the sense and structure of general school education must be directed both to the formation of geometric concepts which are combined with each other and based on experiences and also to pure mathematical or everyday problems. This conception is similar to those of the representatives of the genetic method.

Following this explanation I want to mention the part that computers can take in my conception and how school reacts on the change in our computer society. First, in my opinion, a critical discussion with an extensive view is necessary and secondly, a qualified discussion of course needs and concrete experiences with applications of computers in selected fields. The application of computers in geometry is an example for this.

In the second part of my article I want to show with examples on school geometry how computers can support and enliven geometry teaching and what kinds of objectives should be aimed at. I will complete these reflections with examples of programs in Basic and Pascal as well as some outputs of these. Finally I want to say some words about my experiences with university students.

INTRODUCTION

Let me begin with some general remarks on geometry teaching. Unfortunately, geometry is treated in school like a stepchild though the importance of geometry teaching has been appreciated by many maths-teaching specialists during the last years. This neglect of geometry in school may have been caused for example by two things: first geometry cannot be learned in such a straight forward and simple way as algorithms in computation, and therefore geometry requires thorough knowledge. Second, the problem of developing a systematic course for school geometry remains unsolved. This second handicap can be overcome if we give up the orientation towards a system developed by mathematicians - whether Euclid, Hilbert, Bachmann or anyone else. Rejecting the orientation on mathematical systematics is no big loss, because the task of mathematics teaching at school is not to transmit the results of mathematics research, by which I mean the anticipation of areas of later mathematical studies. On the contrary, we have to look at more general aspects of working with mathematics. With respect to geometry teaching I will specify these general aspects in four points which can be found in the didactic literature (see e.g. Gemener Kongress 1978, Kirsch 1980 and Bender 1983).

1. <u>The relevance of geometry as a means of structuring our three-dimensional world and of using this structure in practice in everyday life, for engineering, the description of nature, etc.</u>

   According to this point of view we have to teach children that mathematics, in this case geometry, can be helpful in solving everyday problems. I elaborated this aspect some years ago in my conception of the so-called "practice-orientated-mathematics teaching" (see Graumann 1976-79). Furthermore, the application of mathematics in science belongs to this point of view. Also, the use of geometry for visualization and as a heuristic medium should be taught to children.

2. The cultural tradition of geometry in our world.
   What is meant here is that all pupils should acquire a basic knowledge of standard geometric concepts and skills. Geometry has also been of great importance in architecture, especially in earlier times, and that sometimes mathematical research has influenced life in general as well as the view of life. Furthermore, the phenomenon that mathematical problems arising from real life problems after a time become indepent, belongs to this point.

3. Geometry as a field for enhancing intellectual competence, as a medium for formal education such as training for creativity, problem solving, argumentation and proving theorems.
   From this point of view geometry represents only one field, but because of its graphic quality as well as the often complex figures and the combinations of concepts, elementary geometry is an indispensable subject for students aiming at intellectual competence at a higher level. Kirsch (1980) states with regard to this point that intellectual training must not mean stolid cramming, it can also bring pleasure and self-confidence. Last I shall mention point 4, which could also be presented in points one or two.

4. Geometry for pleasure and for developing self-confidence.
   From this point of view we could e.g. show students the aesthetics of forms and arrangements, let them work on their own and produce, as well as let them feel their own intellectual power and the autonomy of reasoning.

Keeping these four aims in our mind, it becomes obvious that neither an orientation towards Euclid nor an orientation in modern axiomatism is desirable. Instead for geometry teaching in school, a genetically orientated course with experience in developing geometric concepts and with applications to real life problems or playful pure mathematical problems is necessary. With this kind of geometry teaching, the task of structuring the subjects will not be solved, but we have a wider scope by

choosing the subjects so that the classroom situation and the above mentioned aims can be better realized. On the other hand, in this way we can include geometry lessons better in the mathematics course. Also the special problems, which can be worked out with computers, to be presented later in the article fit in this conception of geometry teaching.

THE USE OF COMPUTERS IN SCHOOL

Before going on, I will briefly deal with the use of computers in school in general. You all, I think, know enough about the role computers play with regard to change in our world. And just as in Germany, you in Finland will certainly find advocates and opponents of the present trend. But global pros and cons do not help us; for on the one hand we cannot stop history, and on the other hand the reason for a predicted negative run always lies in the details. Computers are as all technical innovations not inherently "evil" or "good".

It depends on how we use them. And for that we first have to know a lot about details. The remarkable thing about computers for me seems to be in that they intensify special trends, whether positive or negative. The administration, for example, always contains the danger of seeing individual human beings as objects of administration and produces senseless or even inhuman regulations. Because of the possibility of the multifarious processing of quantities of data and the nation-wide cable-connection this aspect becomes obvious and can become a nation-wide danger. To face such dangers, they must be detected on the basis of expert-knowledge and public discussions, and they must be restricted by courageous actions and legislative limits. Many of the dangers or negative runs could be stopped if enough many people have enough knowledge and experience. Therefore it is our task to reflect on the use of computers in school. I do not want to discuss this general problem here. I only want to say the following: A purely technologically orientated course cannot be the

answer. In addition to some fundamental theoretical knowledge, we especially need knowledge and experience concerning the facilities and consequences of computers in the different fields. A task for maths-teaching specialists, therefore, is to reflect on the sensible use of computers in maths-education. By sensible I mean on the one hand the consideration of the above general objectives, and on the other hand the consideration of the adequacy of the medium computer for the given problem. To elucidate my ideas I shall point out the ways in which computers could be used in geometry teaching, and what kinds of didactic alternatives computers have to offer in geometry education.

COMPUTERS AND GEOMETRY TEACHING

Geometry in this connection is only one of several fields of application where experience about the facilities and limits of computers, as well as their sensible use, could be gathered. Reversely the computer is to be seen as only one of several media that can enrich and enliven geometry teaching. You may ask why I talk about computers in geometry teaching because computers are mostly mentioned in connection with topics which have to do with algorithms. But first of all I like geometry, and secondly what is more important, the new small computers which can make graphics represent a tool, expanding the hitherto very limited possibilities of making many graphics and of carrying out long mathematical operations. Therefore computer-assisted geometry education is more able to make the best of the wealth of elementary geometry so that competencies like "creativity", "flexibility", "differentiated perception", "handling complex problems" can be trained. Especially at the present when routine-problems are increasingly taken care of by machines these general objectives are important. Another advantage of computers in geometry education is the possibility of showing the dynamics in geometric processes and working out systematic variations of geometric subjects. For example you can demonstrate how a circle changes into an ellipse or how the intersection of a plane and a

cube varies. Furthermore, with the modern computers which can make graphics, the students receive visible feedback very quickly so that they can get verification of their effort as well as visible signs of their mistakes. Last but not least with the computer we can produce a lot of aesthetic pictures so that children learn about the aspect of arts and leisure in mathematics or even get motivated for advanced problems.

Before showing you examples of the use of computers in geometry teaching I want to present a survey of geometric themes, which could be supported by computers. I have grouped them the way they are used on the computer. (The following list is, of course, not complete.)

1. The computer-screen-plotter-systems as a medium for the visualization of single geometric figures or sequences of figures
   a) Geometric forms and patterns of polygons, circles or their parts, ornaments and symbols, coverings and special configurations
   b) Curves and families of curves, spirals, rhodoneas, roulettes or dynamical changing conics
   c) Oblique views or three-plane projections of three-dimensional figures
   d) Pairs of figure and image from geometric mappings

2. The computer as a tool for calculations with geometric measures
   a) Linear, square and angular measures with polygons (especially triangles and quadrangles) whereat the polygons are determined in different ways
   b) Linear, square, volume and angle computations with solids like prisms, pyramids, prismatoids and Platonic solids whereat the solids are also determined in different ways
   c) Length, curvature, surface area and volume of curves, respectively, solids of revolution

d) Qualities in analytic geometry such as slope, minimal distances, eccentricity or midpoint

3. Computer procedures as a universal tool to change figures and to test conditions
   a) Procedure for reflecting, rotating, translating, stretching or shearing any plane figure which is given by coordinates
   b) Procedure to produce an oblique view of any three-dimensional figure which is given by coordinates
   c) Procedure to determine the composition of two or more mappings
   d) Procedure to draw a triangle (or quadrangle) which is given by any three (or five) data
   e) Procedure for testing equal length, parallelity and orthogonality of two line segments
   f) Procedure which can produce from a given curve a new one through a fixed technique (such as "pedal curve" or "evolute")

4. The computer as a tool for heuristic research with respect to given problems or finding new questions
   a) Discovering Archimedean coverings
   b) Producing midpoint and diagonal figures for any plane or three-dimensional polygon
   c) Representing simple four-dimensional figures
   d) Producing polygons where a fixed number of linear and angle measures are given
   e) Discovering symmetries in any polygon
   f) Discovering special groups of mapping
   g) Perceiving patterns in ground-plans of historic buildings
   h) Examining the iteration of special mappings (Here I would like to refer to the theory of Mandelbrot and the famous computer patterns with respect to the attractor in the theory of dynamical systems).

In item 4 I have listed a lot of problems which have not been solved. I only wanted to give you an idea of the possibilities. But I think that the use of the computer as a heuristic medium (item 4) should be seen as the most important one. If we could bring at least some of the contents of item 4 into school we would have the opportunity to introduce an idea of what mathematical research really is about.

In the following I shall illustrate some themes by examples.

Example 1: Polygon

The theme "Polygon" going beyond the conventional treatment of triangles and quadrangles is in principle very well suited to the above described conception of geometry teaching, because this theme yields clarification of concepts, manysided applications and stimulations for creativity and aesthetic feeling. The patterns you see on Figure 1 for example are made with a relatively simple program (see program NORM-ECK on Figure 1). With a somewhat extended program where one also can choose the midpoint the patterns on Figure 2 are produced. The correct design reguires a lot of thinkings and computations as to length and angle. Other variations of the first program produce hexagon coverings or regular polygons with their axes of symmetry (see Figure 3) or regular polygons with their diagonals (see Figure 4). The last two programs lead us to questions about axes of symmetry and diagonal-figures of non-regular polygons. We arrive at interesting mathematical problems such as "How many axes of symmetry can a non-regular hexagon have?" or "When is the polygon built from the diagonals similar to the original polygon?".

Furthermore, I shall show you a program with some outputs, which produces a polygon with constant length at each side and an insertable constant angle (see program and outputs on Figure 5). If you want to discuss the different forms of triangles and the shape with regard to the rations of the side, you can also use a simple program (see program and outputs on Figure 6).

Program for REGULAR POLYGONS
with fixed midpoint

```
 1 REM  $ NORM-ECK $

10 LPRINT CHR$(28);CHR$(37)
          (Command to start plotter)
20 LPRINT "O";50;",";-50
          (Command to fix origin)

30 INPUT "Number of vertices";N

40 INPUT "Radius";R
50 INPUT "Beginning direction";A

60 INPUT "Number for colour";C
70 LPRINT "J";C
          (Command to set colour)

80 X0=R*COS(A):Y0=R*SIN(A)
90 LPRINT "M";X0;",";Y0
          (Command to move to first point)

100 FOR I=1 TO N
110 X=R*COS(A+(360/N)*I)
120 Y=R*SIN(A+(360/N)*I)
130 LPRINT "D";,";X;",";Y
          (Command to draw to next point)
140 NEXT I

150 GOTO 80
```

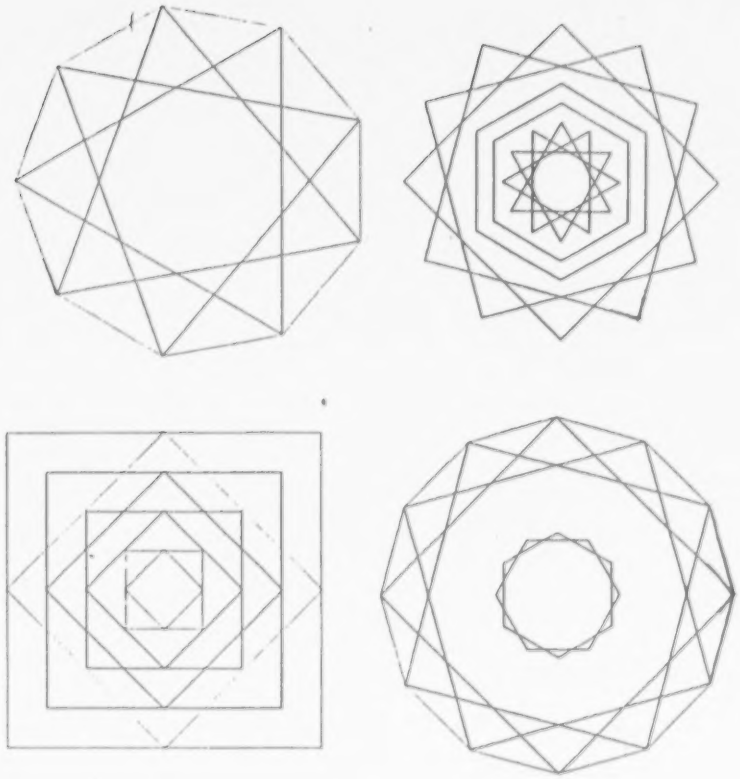


FIGURE 1.

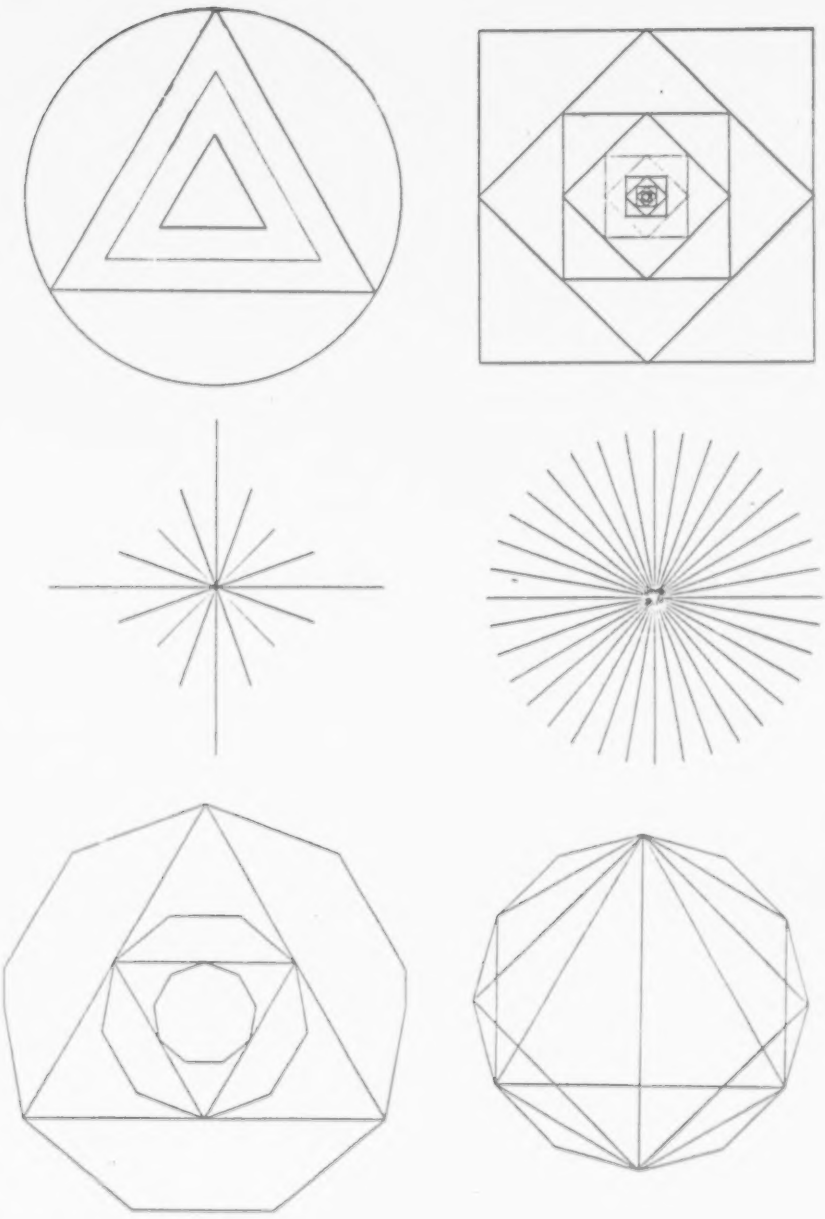


FIGURE 1. (cont.)

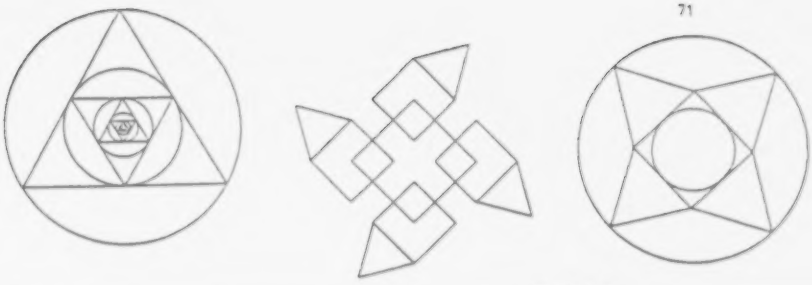


FIGURE 2.

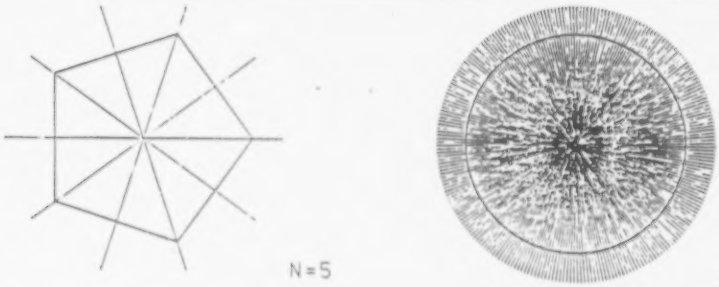
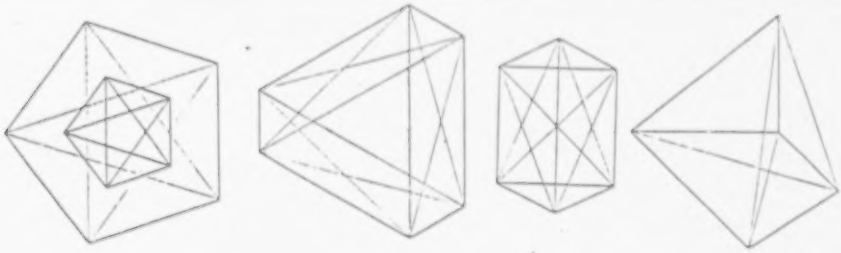


FIGURE 3.



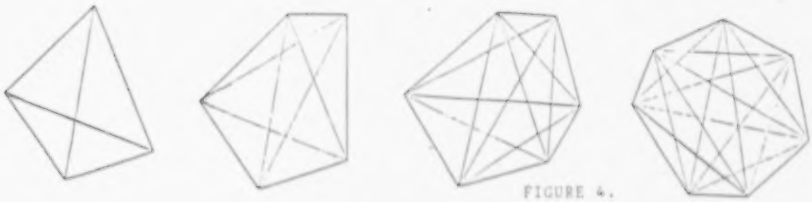


FIGURE 4.

```
 10 REM **ITERATION EINES WINKELS**
 20 LPRINT CHR$(28);CHR$(37)
 30 INPUT "GIB DEN WINKEL EIN !";W
 40 INPUT "GIB DIE ANZAHL DER
          ITERATIONEN AN !";N
 45 IF N=0 THEN 130
 50 INPUT "GIB DEN RADIUS AN ! ";R
 60 INPUT "BEZUEGL. DER X-ACHSE
    GIB DIE ANFANGSRICHGUNG AN !";A
 70 INPUT "WAEHL DIE FARBNUMMER
   #Beachte 0=SW 1=BL 2=GR 3=RT #";F
 80 U=180-W:X=48+R*COSA:Y=-48+R*SINA
 90 LPRINT "J";F;LPRINT "M";X;",";Y
100 FOR I=1 TO N
102 X=48+R*COS(A+I*U)
104 Y=-48+R*SIN(A+I*U)
106 LPRINT "D";",";X;",";Y
108 NEXT I
110 LPRINT "M";0;",";-110
120 GOTO 30
130 LPRINT CHR$(28);CHR#(46):END
```

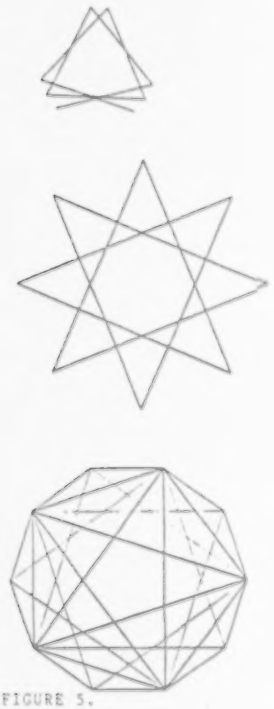


FIGURE 5.

```
 10 REM **DREIECKE**
    #BEI VORGABE DER SEITENLAENGEN#
 15 LPRINT CHR$(28);CHR$(37)
 20 INPUT "GIB DIE LAENGEN EIN!
    #Beachte  a<10  und a<b+c #";A,B,C
 30 INPUT "GIB DIE FARBNUMMER  EIN !
    #Beachte  0=SW 1=BL 2=GR 3=RT #";F
 35 LPRINT "J";F
 40 X=5*(A*A+B*B-C*C)/A
 45 Y=-SQR(100*B*B-X*X)
 50 LPRINT "D";0;",";0;",";10*A;",";0
 55 LPRINT "D";,";X;",";Y;",";0;",";0
 60 LPRINT "M";5*A;",";-3
 65 LPRINT "P";A
 70 LPRINT "M";X/2-1;",";Y/2
 75 LPRINT "P";B
 80 LPRINT "M";X/2+5*A-9;",";Y/2
 85 LPRINT "P";C
 90 LPRINT "M0,-100"
100 GOTO 20
```

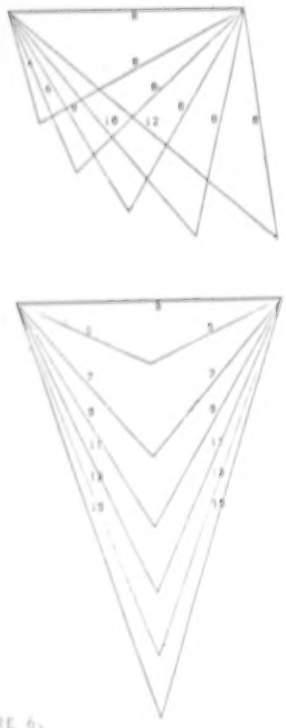


FIGURE 6.

Before showing you examples of the other themes I would like to say something about the use of these programs in school. A prerequisite is that each two or three students have a minicomputer or a terminal of a bigger computer with connectable plotter. I think this will soon be possible in every school because already we can buy minicomputers with a plotter for an affordable price. Computer-languages to be used in schools I would recommend BASIC and later PASCAL; LOGO might also be possible. The students should have basic knowledge in using the special machine (computer) at school and also they should have basic knowledge of coordinates. Other knowledge might be helpful, but it could be acquired during the course. For several problems knowledge about trigonometric functions is necessary. Therefore the teacher should consider introducing these functions earlier than usual. Many programs can be worked out by the students themselves. My opinion is that some programs, especially the easy ones, should be worked out by every student, but also more advanced programs could be used during the course, because in the above sketched conception the point of main effort is the application of programs. Just as in everyday life we use technical instruments we don't know everything about, the students should above all learn to use instruments like computers. Moreover the application of programs introduce occasions for clarification and deepening of geometric concepts and relations. Computers provide opportunities for quick feed-back, and the mistakes can be seen immediately.

Example 2: Circles

With the above program for regular polygons it is easy to demonstrate how the regular polygons go over to a circle. If the basics of trigonometric functions are also given you can show the approach to Pi by the Archimedean method. Furthermore you can use this program to draw figures made of polygons and circles and parts of them (see Figure 7). I think you can imagine what calculations are necessary to produce this or similar patterns with the program.

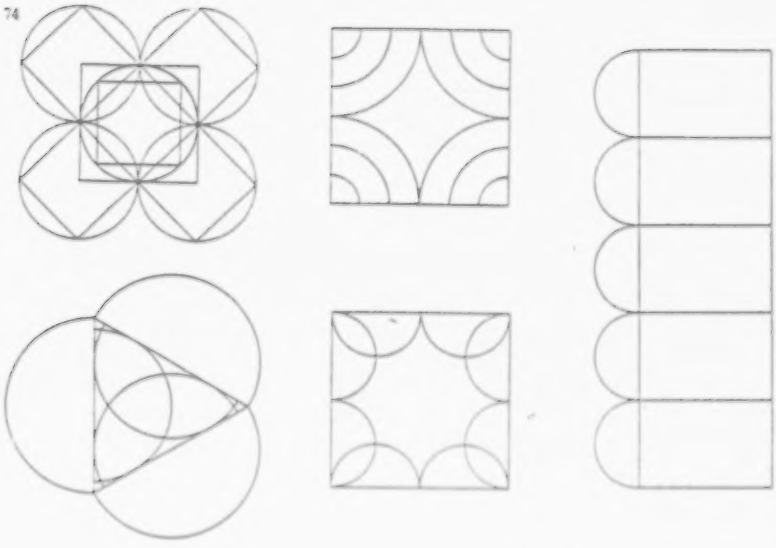


FIGURE 7.

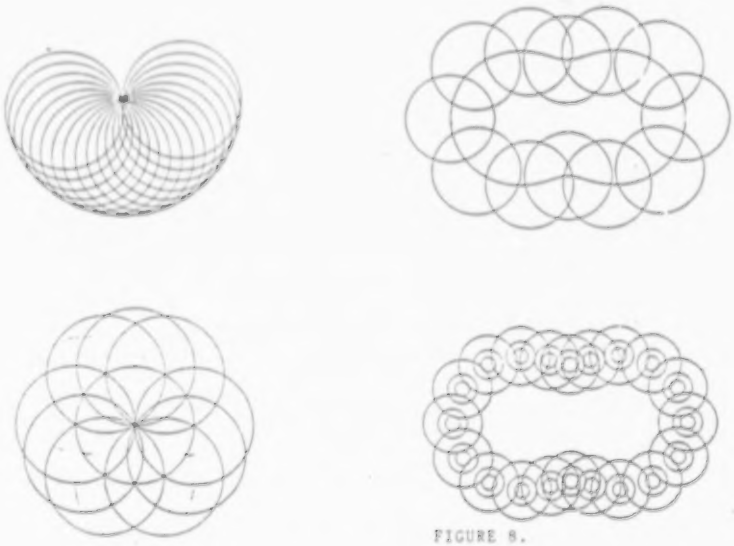


FIGURE 8.

A program where the plotter draws circles whose midpoints lie on a circle or any curve was made by students at the university. Very nice patterns can be produced with it (see Figure 8). These pictures enable students e.g. to calculate with angles, to ask questions about factors of 360 or the iteration of irrational angle measures. Questions about the square measure of intersections of circles could also be dealt with in this context.

Example 3: Solids

After the regular polygon it would be good to deal with regular polyhedrons. Because drawings of these figures are not easily done, a procedure for drawing regular polyhedrons in any position can help. With questions about rotations of such a solid before plotting and about points, segments and surfaces in a special position the perception of space could be practised. In particular with the cube more operations could be made than for example intersections with planes. A computer program which delivers many pictures in a short time is therefore a good help.

The cube is already known by younger pupils. In connection with geometric puzzles you can treat them with the question for all positions of one cube. After they have made several pictures on their own to get an understanding you can let them use a program which frees them from drawing and lets them concentrate on the main point of the problem. By the way, I am not in favour of freeing the pupils from doing drawings, but in some cases it is advantageous to use an instrument such as a computer.

A computer provides helpful assistance for computations of surfaces and volumes of special solids. The computer provides an opportunity to work out problems, which are not merely dull computations. For example you can work out a program on quadratic pyramids where any two of the four data: length of the basis, height, surface, volume are given. The others will be computed. You can also introduce the equation of third degree, which gives you an opportunity to talk about the history of mathematics and the formula of Cardano.

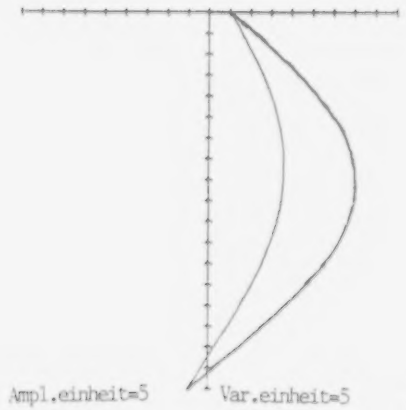
Example 4: Mappings and Curves

A student of mine made an extensive program on geometric mappings, i.e. on rotations, reflections, central stretchings, axial stretchings and shearings. The program on the one hand comments on the mappings and on the other hand draws the original and the image of any wanted polygon.

While discussing different functions it is good if the students can get a lot of graphic representations of the various functions. Also you approach new aspects in this theme, if you give visual presentations and afterwards discuss whole families of functions. For this you may use a program which plots graphs of every function (see Figure 9).

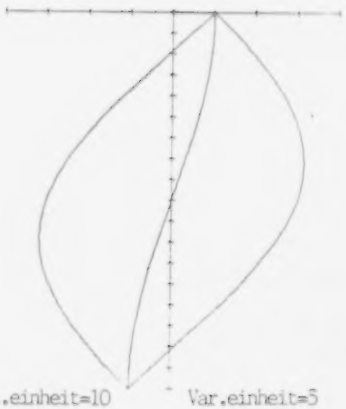
With other curves like spirals, rhodoneas or roulettes the viewpoint of the students can be extended. Even some ready made programs give the possibility to become acquainted with such types of curves that have never been introduced in schools. Besides you can produce interesting and nice ornaments with curves, if you can draw them as easily as a computer can (see Figure 10).

The thoughts and examples presented here mark just a beginning. Many experiments must still be made. Teacher training in this area must be organized. According to the concepts introduced here one should not try to plan a systematic course. On the contrary the course should be similar to the courses in school. The task of teacher training is to accumulate experiences with the new medium, computer, on the field of elementary geometry and to reflect in a didactical way on the experiences. I, for example, did a course with students where they first had to make programs about polygons, circles and rhodoneas, similar to those I have shown. During the course the students got acquainted with the computer's graphic system and could also reflect on mathematical problems students at school will have by using such programs. We arranged five groups of two to four students and each group had to work on a bigger project. The task for each group



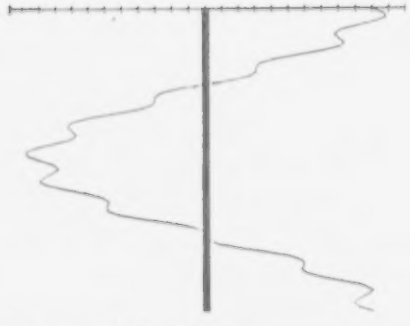
Ampl.einheit=5 Var.einheit=5

F=COS(10·V)+7·SIN(10·V)
F=COS(10·V)+3.5 SIN(10·V)

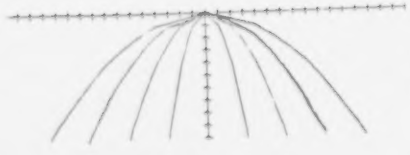


Ampl.einheit=10 Var.einheit=5

F=1·COS(10·V)+0·SIN(10·V)
F=1·COS(10·V)+3·SIN(10·V)
F=1·COS(10·V)+(-3)·SIN(10·V)



F=SIN(V·10)+10·COS(V)



Ampl.einheit=6 Var.einheit=3

F=1·RND+5·SIN(10·V)

FIGURE 9.

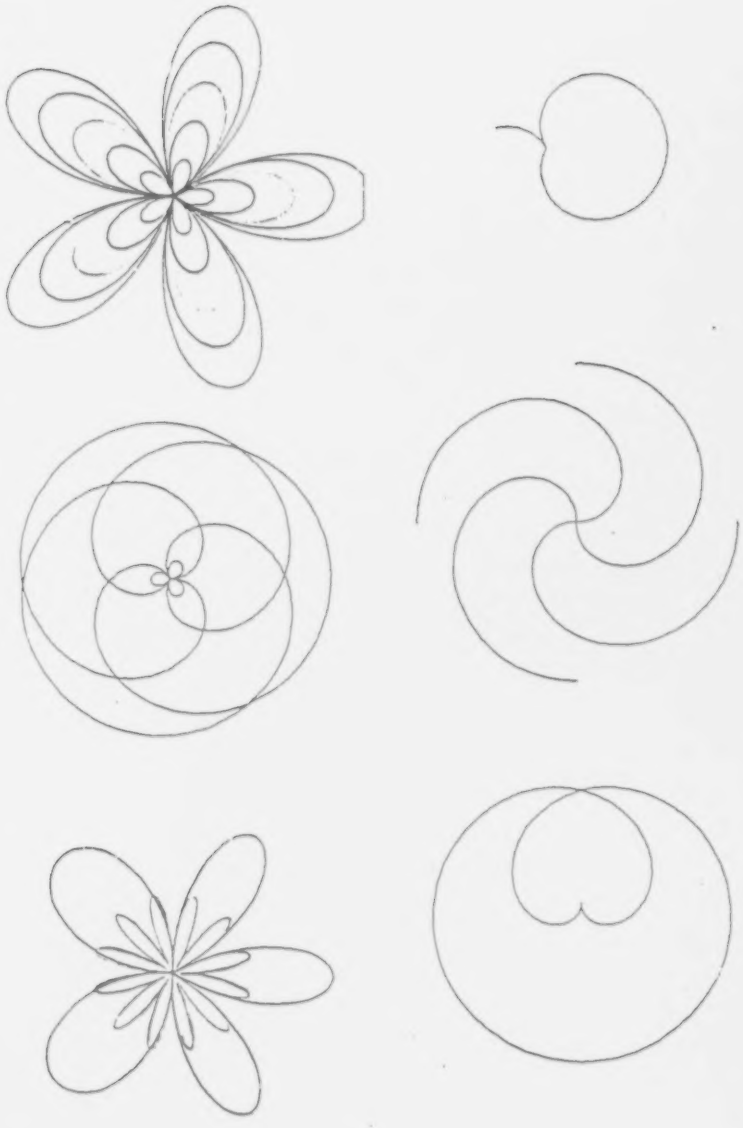


FIGURE 10.

was to make a program and consider the use of this program in school. The project themes were "mappings and symmetry of non-regular polygons", "Measures of pyramids and parallelepipeds", "Curves in Cartesian coordinates" and "Constellation of the planets at one's birthday". We learned during this course that the programming of such projects takes a lot of time and requires a lot of knowledge of computer-science. Therefore such projects can be done in school only in special study groups. But if you use good programs all students can work out interesting applications and will receive motivating experiences.

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FROM PROBLEM SOLVING TO PROBLEM FINDING IN MATHEMATICS INSTRUCTION

Bernd Zimmermann

ABSTRACT

In the 1980's the focus of school mathematics is on problem solving (Agenda for action, NCTM 1980). The fact that problem solving means more than solving verbal problems and, furthermore that it should be extended into problem finding is the main concern of this paper. After discussing some possible reasons for problem solving and finding the article presents examples by which the reader might use the opportunity to apply the method of "discovery learning". The experiences might provide some help in judging the use of the "principles of problem solving and finding". The paper ends with some teaching suggestions for problem solving and finding.

WHY PROBLEM SOLVING OR FINDING

Let us first think of some reasons, why problem solving should become a leading idea during a mathematics lesson:

1) There are mathematicians like Polya (1980) and Halmos (1980) who consider problem solving the heart of mathematics. There should be more emphasis on the process of doing mathematics than on more learning of mathematical results (cf. Lakatos 1976).

2) The attempt to introduce "New Maths" into the mathematics curriculum was in many countries followed by unfavourable experiences. Taking into account the many reasons for this situation it was found out that the main goal - more under-

standing of mathematics - very often was not attained by the students. Furthermore a severe decline in routine skills was stated (Kline 1973).

3) On the other hand the idea of "back to basics" - which can be understood as a reaction to the bad experiences with "New Maths" - was not satisfactory either. An extensive study of the mathematical knowledge of students in the USA gave the following result: "While students displayed a fairly high level of skill in wholenumber computation in the assessment, their abilities to apply these skills to the solution of realistic problems were significantly lower." (Hill 1980).

4) Taking into account these negative results the NCTM (National Council of Teachers of Mathematics in the USA) came to the conclusion, "...that problem solving be the focus of school mathematics in the 80's" (NCTM 1980). This statement was strongly supported by the outcome of a comprehensive survey on the opinions of people involved in the teaching of mathematics in the USA (Priorities in School Mathematics"-project, NCTM 1981). 88 % of all persons who returned the questionnaire (some 3000 out of 10 000) gave the first priority to problem solving.

5) Complex problems of modern society like environmental problems require more connected and connecting thinking. For this need a problem solving approach might prove useful as well (cf. Vester 1983, 1984).

6) Theoretical models of mathematical Problem solving - checked by simulation on computers (representing "artificial intelligence" - AI) - support the importance of general and specific strategies as well as a comprehensive knowledge base ("expert-systems", cf. Lenat 1984). E.g. Lenat (1977) developed a computer program ("Automated Theory Formation in Mathematics"), which was able to find mathematical concepts (like "perfect numbers") and conjectures (like Goldbach's confec-

ture). Furthermore, research results in the field of AI emphasize the importance of automatized routine learning as well (cf. (3)) as a necessary condition for creative Problem solving. For example a whole algorithm can be "chunked" (cf. Krutestkii 1976) and taken to the working memory as one unit so that memory can be relieved from thinking about each single computational step. In this way memory is free for higher order creative activities (cf. Wachsmuth 1985).

There are many reasons why we should place more emphasis on problem solving during mathematics lessons. But one should stress problem finding as well! This theme has been neglected at nearly all levels of education until now.

Some reasons why problem finding is essential are the followings (cf. Brown & Walter 1983):

1) Very often the creation of a new problem can be an important strategy for solving a given problem.

2) If you have already solved a given problem, an additional question or problem can support a deeper understanding of the original problem.

3) Perhaps one reason for boredom in the classroom might be the fact that very often answers are given to questions the students never asked. So, finding their own questions, could stimulte the students and foster their creativity.

4) Furthermore, "mathophobia" might be reduced by the fact that new questions of problems cannot be judged by the teacher by simple "right" or "wrong".

EXAMPLES

Organisation of this section

To give some concrete ideas about a mathematics lesson oriented towards problem solving or finding some examples of mathematical problems or situations would seem very important.

The four examples are presented according to the following principles:

1) The examples should lead from more closed situations to more open ones.

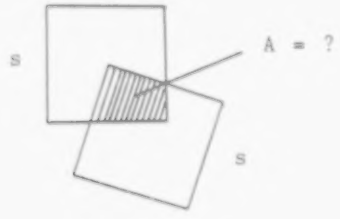
2) As far as possible the problems should be analyzed in the following way:
   a) Reference to the four main strategies "understanding the problem", "devising a plan", "carrying out the plan" and "looking back" (and ahead) of Polya (1973).
   b) The main strategies in feasible solutions should be worked out.
   c) Possible connections with the Finnish syllabus (Kouluhallitus 1981, 1982) shall be sought.

3) The reader can stop reading at the marked places (or others) and can deal with the problem in his own way.

Example 1

The Problem

Two squares, each of length s on a side, are placed so that the corner of the one square lies in the centre of the other, as shown.



Describe in terms of s the range of possible areas representing the intersection of the two squares, (Schoenfeld 1980).

** Stop reading here and try to solve the problem! **

Solution of example 1

1) Understanding the problem

   We have to find a formula A(s)=? for any location of the squares.

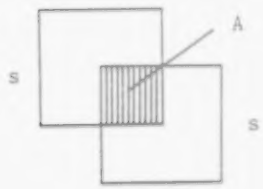
2) Divising a plan

   To represent the arbitrary location of the squares - taking into account the specific "boundary-conditions" - it might be useful to restate the problem in the following concrete way: Imagine having two quadrangle beer mats. You place one of the mats in the centre of the other and turn it around this point. The shape of the intersection changes all the time.

** Stop reading and make your own observations. **

It might happen, that you were struck by the observation of the following special case:

(1)

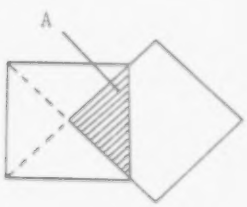


In this case the following formula is obvious:

$$A=s^2/4$$

If you keep on turning the other mat, you perhaps notice the following simple special case:

(2)



As the intersection is one of four obviously congruent parts of the mat the area must also be:

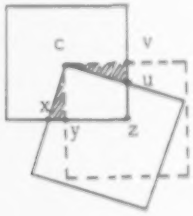
$$A=s^2/4.$$

Thus one might think that this formula is independent of the location of the squares.
Perhaps it is possible to derive this general conjecture from one of our special cases!

** Stop reading and try to carry out the plan! **

3) Carrying out the plan

If we "superpose" the first picture (representing the "general case") and the picture of the special case (1), we get the following figure:



The triangles XYC and UVC are congruent (ASA) and, therefore, have the same area. Thus the same must be true of the quadrangle XZUC and YZVC.

Which means that

$$A=s^2/4$$

holds for any situation.

4) Looking back and ahead

You can try to deal with several generalized or analogous problems:

a) How many ways are there to divide a square into four congruent parts?

b) What about the volume of the intersection of two cubes placed in an analogous way as the two squares?

c) What would happen if the two squares had different sizes? (Cf. Penttilä et al. 1985).

d) What would happen if you try to generalize the problem to arbitrary regular polygons?

Strategies used in example 1:

The following strategies were important in the solution presented here:

- making diagrams,
- restating the problem by using two beer mats (or two squares on two different transparencies),
- examining the special cases (most important for this problem),
- deriving a conjecture from the special cases,
- deriving the general from a special case,
- decomposing and composing areas.

Connecting example 1 to the Finnish syllabus

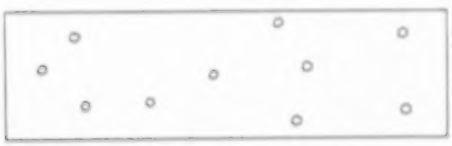
- The mathematical content incorporated in this problem is "congruence" and "computation of areas".
- This content should be treated on the 8th and 9th grades of the Finnish comprehensive school (Kouluhallitus 1982).
- Possible applications of this example might be to motivate the use of the congruence theorem ASA or practising this theorem as well as computing areas.

Example 2

The problem

A window maker needed assorted triangular pieces of glass for a project.

A rectangular sheet of glass that he planned to cut had ten air bubbles, no three bubbles in a row:



To avoid having an air bubble in his finished project, he decided to cut triangular pieces by making the air bubbles and the corners of the plate the vertices of a triangle. How many triangular pieces did he cut (cf. Billstein, Libeskind & Lott 1981)

** Stop reading and try to solve the problem! **

Solution of example 2

1) Understanding the problem

   If we draw different decompositions of the sheet into triangles, we always get 22 pieces. Are there exceptions, or can we prove that there are always 22 triangles?

** Stop reading and try to solve the problem now! **

2) Devising a plan

   Surely the number of triangles should depend on the number of bubbles. Two ways seem reasonable:

   a) We study some simple special cases (1;2;3;4;...;10 bubbles) and search for an inductive pattern.

   b) If the number of triangles should depend only on the number of bubbles, we should look for some invariant properties of triangles and bubbles (points) for different decompositions (assuming our conjecture is correct this means working backwards).

** Stop reading and try to solve the problem now! **

3) Carrying out the plan

   a) Let b denote the number of bubbles and t the number of triangles, respectively. In the way described above we get the following table:

b	t
1	4
2	6
3	8
4	10
.	.
.	.
.	.
9	20
10	22

The following conjecture is evident:

$$t=2\cdot b+2 \quad (b=10 \Rightarrow t=22\ !)$$

This can be proved by induction bearing in mind that an additional bubble would cause a single triangle to be substituted by three new ones.

b) The sum of the measures of the angles in a triangle is 180° and independent from the decomposition of the sheet. The sum S of the measures of the angles formed by all triangles is, therefore, always 180° times the number of triangles. As the angles of the triangles are either angles around the air bubbles or corners of the rectangular sheet we also have:

S= sum of all measures of angles around all bubbles + sum of the measures of the four angles at the corners of the rectangular sheet.

$$S=360^\circ\cdot 10+90^\circ\cdot 4 = 3600^\circ+360 = 3960$$

Consequently the number of triangles is

$$3960^\circ:180^\circ = 22.$$

4) Looking back and ahead

We can check our solution by some other examples (number of bubbles). There are at least the following possibilities for generalization:

- take m air bubbles,
- take an arbitrary n-gon instead of the rectangular sheet.

(Result: number of triangles = ((n-2) 180 +m 360):180 = n-2+m 2)

What would happen, if we wanted to get k-gons as glass pieces? What about decomposing a quader into polyhedrons with the same number of faces?

Strategies used in example 2

- Restating the problem by results of practical experiences is important in making the problem solving process more goal-oriented.
- Looking for an inductive pattern by examining special cases is a very helpful approach, which leads to a solution of the problem.
- Working backwards and looking for invariants was a constructive alternative, which lead to the solution as well.

Connecting example 2 to the Finnish syllabus

- The mathematical content incorporated in this problem is angulars and angular theorems.
- This content should be treated on the 7th and 8th grade of the Finnish comprehensive school (Koululaitus 1982).
- Possible applications of this example might be practice with angular theorems.

2) Devising a plan

a) Trial and error: take some short roads and check, whether you can meet the conditions of the problem. If you have enough roads, add their distances, make another trial and compare the sums. How can we be sure of finding the shortest sum in this way?

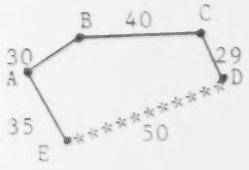
** Stop reading and try to solve the problem now! **

b) Examine special cases:

(i) When we made our first trials, we perhaps checked the following special cases:

In these cases it is not possible to leave a single road covered with snow: you have to clear them all!

If we have "loops" the situation is different:

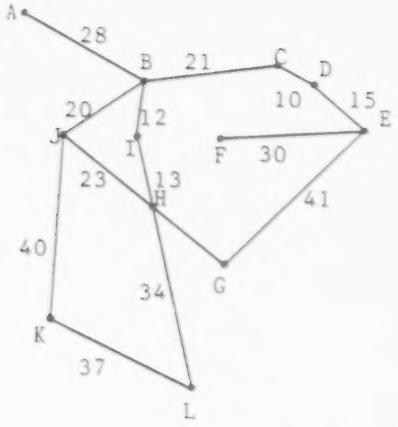


Obviously you can leave the longest road covered with snow.

(ii) Let us look for very short roads and start in an arbitrary town. Only the short roads should be cleared.

Example 3

The Problem



The figure presents a road map (towns A - L, distances in kms). The roads are covered with snow. The road net is to be cleared of snow so that all towns are connected and the total length of all roads to be cleared should be as short as possible (cf. Kiesswetter & Rosenkranz 1982).

** Stop reading and try to solve the problem! **

Solution of example 3

1) Understanding the problem

First of all, one has to talk over many practical questions in the classroom: Is the number of snow-ploughs important, and should we know where they are kept? Should we take into account the trips made on roads which have already been cleared? Is it necessary to drive each road at least twice, because one has to clear the road in both directions? Perhaps we should first make simple assumptions to make the problem less complex!

** Stop reading and try to solve the problem now! **

** Stop reading and try to solve the problem now! **

3) Carrying out the plan

a) Let us look for "loops" and leave out the longest road (not to be cleared) in each loop. Continue as long as there still are loops. The result is a "minimal spanning tree", representing the shortest net of roads to be cleared.

b) We assume that we are a snow-plough driver, who has lost his road map and who starts his tour in an arbitrary town. He can only see the road signs at the end of his town, which indicate the distances to the next towns. It seems reasonable to clear the shortest road to the next town first. After arriving here, he will look again for the shortest road to a town to which there is still no connection. Now he can think of the whole system of already cleared roads with the corresponding towns as a new "superpoint" or "supertown" in the remaining net. This way of combining several smaller units into a larger unit is called "chunking" by Krutetskii (1976) (cf. p. 83). Our driver has to look again (and clear) only the shortest road going from this supertown to a new town, taking into account the possibility of driving back through this supertown (on already cleared roads). He has to continue this process as long as there still are towns without a connection.

4) Check and compare the results of the two different algorithms in (i) and (ii). They should lead to the same remaining net! Choose other starting points and loops as well as other road nets. What about a proof or computer programs? What about a similar problem in another area of application (e.g. telephone networks etc.)?

Strategies used in example 3

- Trial and error is the first sensible approach. Very often this strategy leads to materials or observations, which help

in finding a solution for the problem. Kilpatrick (1967) found that in a special sample of 14-year-old children the ones who applied the trial and error approach were more successful than the ones who applied algebraic techniques.

- Restating the problem is important especially in determining those conditions of the problem, which are less relevant or should be omitted in the first attempt.
- "Examine special cases" (road nets) seems to be the most fruitful strategy for this problem.
- "Chunking" is important in a) when we studied "loops" (concept formation) as well as in b) when we established "supertowns" (cf. pp. 82-83, Krutetskii 1976).

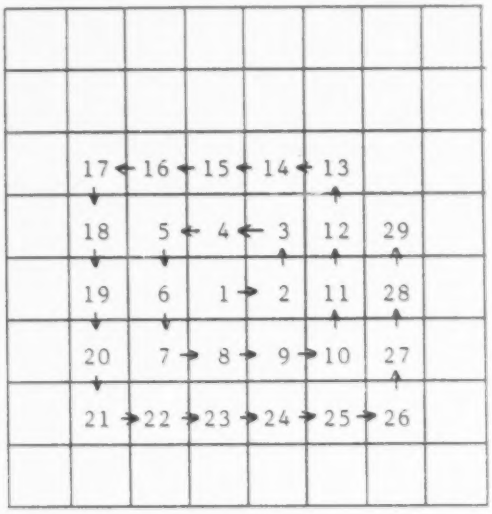
Connecting example 3 to the Finnish syllabus

- At a first glance there seems to be no connection to the Finnish syllabus. Graph theory is not a content treated in Finnish mathematics lessons. The same is true in West Germany. At least two possibilities are feasible:
  a) The example does not fit into the syllabus and, therefore, should be omitted.
  b) Some graph theory should be implemented into the normal curriculum. The last suggestion could be supported e.g. as follows: graph theory has at least as many applications as analysis. Examples are: linear programming, the travelling salesman problem, traffic regulation and construction of telephone networks. Furthermore to deal with problems of graph theory, only a very small amount of mathematical knowledge is necessary. To solve complew problems in analysis, a tremendous amount of algebraic techniques is needed. Problems of graph theory might sometimes provide a better chance for the student to train in creativity and to succeed (cf. the Koenigsberg bridge problem).
- This example can be chosen to demonstrate that extreme value problems can also be solved by other methods than those of analysis. Extreme value problems should be treated in the Finnish upper secondary school (cf. Kouluhallitus 1981).

Example 4

The situation

Let us "wind up" the natural numbers into the the following spiral (Ulam's spiral, cf. Brown & Walter 1983):



** Stop reading. Can you see a pattern? Would you like to ask some questions? You can continue to draw the spiral as far as you want! **

This situation was selected to give a chance of finding your own problems or questions. A few questions are presented here:

1) Which numbers are on the diagonals passing through 1?
2) Which numbers are in the rows or in the columns?
3) Where has the number 87 (n) to be placed?
4) How are special numbers (e.g. odd-, even- or prime numbers) distributed?
5) What happens if you change the structure of the spiral?
6) How has one to change the structure of the spiral to meet a given distrubution of special numbers?

If you wish you can look up the answers to these or to your own questions. After finding a solution you might think about the strategies you applied in your problem solving process.

Connecting example 4 to the Finnish syllabus

Properties of natural numbers should be a theme of mathematics lessons in Finland at the beginning of grade five. Explicit training of the student's creativity is possible by use of this material, as well as some practice in concept formation in the area of natural numbers.

SOME PRINCIPLES OF PROBLEM SOLVING AND FINDING

Some important strategies

The four steps in the problem solving process and most of the strategies used in the examples were first mentioned in this form by Polya (1973), who has presented them in his well-known list of strategies. This list has been completed by Wickelgren (1974) and Schoenfeld (1980). Or course all of these strategies have not the same importance in mathematical problem solving processes. We want to present some of the most important ones.

- Symmetry is a strategy, which is not only important in geometry and game theory but also in physics (geometrical optics) and arts. You can find principles of symmetry in biology as well (cf. Zimmermann 1983).
- Mathematical modelling is the strategy one has to use in solving problems from applied mathematics (cf. example 3):
  a) A more or less real life situation is given.
  b) A mathematical model, which meets the needs of the problem as well as possible, has to be developed. Normally, one has to simplify the problem to reduce its complexity.
  c) One tries to solve the problem represented by this model by suitable mathematical methods.

d) Going back to the level of reality to check and interpret the solution gained in c) with respect to the real "boundary conditions" will close this "loop".

- Problem solving by induction is a very important strategy not only in the very narrow sense of mathematical induction. Growth of knowledge very often follows the inductive way, which goes far beyond formal induction. For example mathematical induction proved that formal logic (including the induction principle) cannot prove everything (cf. Goedels incompletness theorem, Goedel, 1931). Therefore, informal (inductive) methods like those described by Lakatos (1976) should be applied to help fostering the process of recognition in science. In this way, we have not only psychological but also logical reasons to support inductive thinking in school children.
- Changing the mode of representation very often brings a problem into a form, which makes a solution easier. E.g. representing the map of Koenigsberg by a graph helps to see that a closed tour cannot be drawn by one single line.
- In this or some other way an analogous problem is very often created, which makes a solution easier (cf. Ploya 1973).
- The importance and power of "chunking" had already been described (pp. 89-90).

One has to remember that strategies like the ones described here very often have a large range of overlapping or denote activities on different levels of generality.

Strategies for selecting strategies (metastrategies)

A student, who has already learned something about strategies like those mentioned by Polya, has, nevertheless, in a concrete situation of solving a problem the following difficulty. How can he know which of these strategies is appropriate and should be selected in the different stages of the problem solving process? The knowledge of some strategies for selecting strategies

(metastrategies) might be useful (cf. Schoenfeld 1980). Two examples are given here:

- Look for strategy-specific clues: For example, if there is - more or less explicitly - a statement like "n N" in the Problem, mathematical induction might be suitable. If you find words like "not" or "exist" in the problem text, you might try proof by contradiction. If you want to prove an equation or inequality, working backwards or examining special cases might be helpful.
- Think about special strategies, which are typical of specific areas. For example, symmetry is often useful in geometry and game theory, induction for number theory.

Strategies for creating new problems

- "Be smart - generalize" is quite a well-known strategy in mathematics for finding new problems, and it does not give too much concret help.
- Asking "what-if-not"-questions in a systematic way (cf. Brown & Walter, 1983) might often lead in many times to new problems. If you look at the questions b), c) and d) (p. 87) could be derived in this way.
- In the same section just refered to, question a) can be chosen as a more distant but analogous problem. Analogy is thus not only an important strategy in problem solving but also in problem finding.
- Changing the field of application of a (already solved) problem might lead to further questions as well. This was demonstrated in item 4) (p. 94).
- Of course we could after some experience also generalize some methods of generalization: going from plane to space (from lower to higher dimensions) is a very usual strategy. Enlarging the number of objects or variables in a problem is another example of a very general method in creating new problems.

- The last monor but nevertheless important suggestion for problem finding is: play with situations and material!

Teaching problem solving

The following recommendations are neither very new nor comprehensive. Nevertheless, making them known might be of some help for the mathematics teacher (cf. Suydam 1980):

- "Teaching for problem solving is, in the beginning, an uphill battle. Teachers meet insecurity, apathy, and frustration even in the better students. But the rewards are worth the effort." (Kantowski 1980). Therefore, be patient with your students!
- Let the students have their own experiences. Let them not only find solutions to problems and create new problems, but also strategies.
- Let the students become conscious of their problem solving strategies, for example by talking about their different solutions or difficulties.
- Take strategies seriously. Let the student write the strategies they have found in a special exercise book.
- Let the students make conjectures and be tolerant with their mistakes.
- Tell them that "good mathematics" is not mainly defined by a lack of mistakes, but by the quality of ideas.
- Don't use highly sophisticated formal language. Bear in mind that the student, the language and mathematics should have an opportunity to grow steadily and at an appropriate rate in the mathematics classroom.
- Be careful in selecting the material and problems for your students. Every student should have successful experiences.
- In the Heinola seminar on mathematical problem solving in January 1986 there was a discussion on the question whether the teacher should place more emphasis on the teaching of strategies or free problem solving without thinking of strategies. An answer to this question might depend on the situation

in the classroom. The teacher has to judge the amount of guidance his students need.

- Don't forget to compliment your students!

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WIE KANN DER LEHRER LERNSCHWACHEN SCHÜLERN HELFEN?

Leila Pehkonen

ZUSAMMENFASSUNG

In der Untersuchung, die im Frühjahr 1982 durchgeführt wurde, wird das Niveau des Beherrschens der Grundrechenarten bei 13jährigen lernschwachen Schülern untersucht. Zu Beginn wurden die Versuchspersonen (N=19) mit einem Test aus 40 Aufgaben getestet. Danach wurden die Fehlertypen aller Schüler untersucht. Die Fehler wurden als systematische, Zufalls- und Konzentrationsfehler klassifiziert. Auf Grund der Testergebnisse wurde ein Leistungsprofil für jeden Schüler erstellt. Nach der Fehleranalyse und dem Profilieren wurden aus den Versuchspersonen die fünf Schwächsten gewählt, deren Leistungsprozess mit Hilfe der Technik "lautes Denken" untersucht wurde.

ZIELE DER UNTERSUCHUNG

Die Forschungen über Schulergebnisse helfen oft vor allem Lehrplanforschern und Schulbehörden. Das in umfangreichen Forschungen gewonnene durchschnittliche Wissen ist zur Planung grosser Linien notwendig. Doch wenn der Lehrer mit den Schwierigkeiten eines Schülers kämpft, ist er nicht daran interessiert, was die Schüler durchschnittlich leisten können. Um ihm helfen zu können, muss er wissen, was genau dieser Schüler kann.

Das erste Ziel der Untersuchung bestand darin, das Niveau des Beherrschens der Grundrechenarten bei 13jährigen lernschwachen Schülern zu untersuchen. Das zweite Ziel war, Hinweise darüber zu erhalten, wie man den Förderunterricht effektiv gestalten kann. Die vorliegende Untersuchung will keine allgemeingültigen

Ergebnisse erzielen, sondern nur detailliertes Wissen über die Kenntnisse einer Unterrichtsgruppe und besonders von einigen lernschwachen Schülern erhalten.

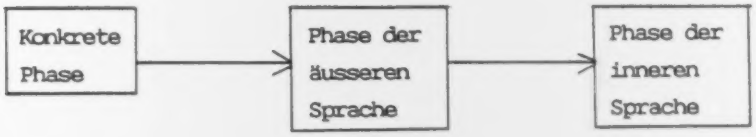
DER THEORETISCHE HINTERGRUND

Der theoretische Bezugsrahmen dieser Arbeit ist die Lerntheorie des sowjetischen Psychologen P.J. Galperin (siehe Galperin 1969, 249-272). Galperin unterscheidet verschiedene Phasen beim Lernen geistiger Handlungen. Es ist erst dann möglich die folgende Phase zu erreichen, wenn man die Handlung in der vorigen Phase beherrscht. Beim Lernen geistiger Handlungen kann man nach Galperin fünf verschiedene Niveaus oder Phasen unterscheiden:

1. Dem Schüler wird eine vollwertige Orientierungsgrundlage der Handlung gegeben.
2. Die geistige Handlung zeigt sich zunächst als materielle Handlung, also als aktive äussere Handlung am Lerngegenstand.
3. Der Lernende soll nun keine äussere Handlung mit dem Lerngegenstand mehr durchführen, sondern diese nur sprachlich beschreiben.
4. Der Lernende soll die Merkmale und Bedingungen des Handlungsverlaufs für sich sprachlich formulieren.
5. Die gesamte Handlung wird nur noch auf der inneren Ebene durchgeführt.

Oft reduziert man diese fünf Phasen auf drei (Fig. 1). So habe ich es auch bei meiner Arbeit gemacht, wobei die erste und zweite sowie die dritte und vierte Phase zusammengefasst wurden. In der konkreten oder materiellen Phase darf der Schüler mit konkreten Gegenständen handeln. Eine Handlung in dieser Phase ist erst dann vollständig, wenn der Schüler genügend mit Hilfe konkreter Gegenstände arbeiten darf. In der folgenden Phase, der äusseren Sprache, lässt man nach und nach das konkrete Material weg und beginnt die äussere Sprache als Hilfsmittel des Denkens zu benutzen. Der Schüler denkt laut, während er arbeitet. In der

dritten Phase der "inneren Sprache" erfolgt die Handlung ohne äussere Hilfsmittel, nur mit Hilfe der inneren Sprache, des Denkens.



FIGUR 1.

Nach Galperin wird es nicht möglich sein, neue geistige Handlungen ohne eine konkrete, materialisierte Grundlage zu bilden, das gilt besonders auch für Kinder.

In der Praxis bedeutet die Galperinsche Lerntheorie, dass die Schüler mit konkretem Material arbeiten müssen, wenn sie eine neue Handlung lernen. Es ist nicht genug, dass der Lehrer vor der Klasse demonstriert oder an die Tafel zeichnet, sondern die Schüler sollen selbst mit Material arbeiten. Nach dieser materialisierten Phase kommt die Phase der äusseren Sprache. Die Schüler sollen wieder selbst die äussere Sprache benutzen. Offensichtlich ist es so, dass im Schulunterricht zu schnell direkt auf die Ebene der "inneren Sprache" übergangen wird. Oft zeigt sich, dass einige der Schüler noch auf einer niedrigeren Ebene sind. Ihr Lernen verlangt noch konkrete oder gesprochene Handlung. Auch der Erwachsene denkt laut, wenn er etwas macht, was ihm nicht bekannt ist und die dazuhörige Handlung noch nicht automatisiert ist.

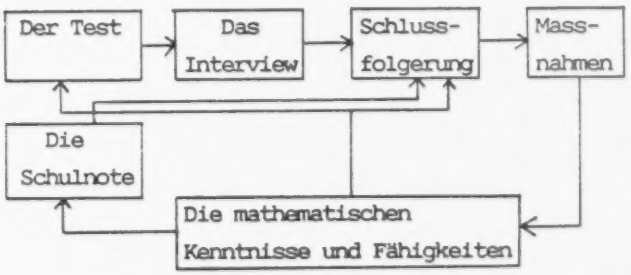
VERSUCHSPLAN

Die Abbildung (Fig. 2) stellt den Versuchsplan dar. Den Ausgangspunkt bilden die Mathematischen Kenntnisse und Fähigkeiten

des Schülers. Dazu gehören auch die Beherrschung der Grundrechenarten und der schriftlichen Rechenverfahren mit natürlichen Zahlen.

Das Lernen der schriftlichen Rechenverfahren steht in Zusammenhang mit der Automatisierung der Handlung. Nach der Galperinschen Lerntheorie ist die letzte Phase die sprachliche Verinnerlichung (die innere Sprache). Die Anzahl der Routineaufgaben nimmt in den späteren Schuljahren im MU zwar ab, aber trotz allem braucht man im 7. Schuljahr schriftliche Rechenverfahren, um mit mehrstelligen Zahlen zu rechnen.

Die Grundrechenarten und schriftlichen Rechenverfahren werden mit einem Test von 40 Aufgaben getestet (siehe Anhang). Der Zusammenhang von Note und Testergebnis ist allerdings als gering anzusehen, weil die Schüler im 7. Schuljahr nicht so viel Grundrechenarten mit natürlichen Zahlen durchführen. Auf Grund des Testes wurden die fünf schwächsten Schüler für ein Interview ausgewählt, in dem die Technik des "lauten Denkens" angewandt wurde. Ausgehend von der Lerntheorie Galperins, habe ich versucht, durch Kombination von schriftlichem Test und mündlichem Interview mit "lautem Denken" herauszufinden, welche geistige Ebene schwächere Schüler im Rechnen tatsächlich erreicht haben. Auf Grund der Ergebnisse kann man dem Lehrer Hinweise für Fördermassnahmen geben.



FIGUR 2.

FRAGESTELLUNG DER UNTERSUCHUNG

Die Untersuchung sollte Antwort auf folgende Fragen geben.

1. Ein wie grosser Anteil der Schülerfehler in den Grundrechenarten ist systematisch, und welcher Anteil sind Zufallsfehler? Auf Grund der früheren Forschungen kann man annehmen, dass Zufallsfehler zahlreicher als systematische Fehler sind (z.B. Cox 1975).

2. Was für systematische Fehler machen die Schüler bei schriftlichen Rechenverfahren?

3. Wie ändert sich die Fehlerverteilung bei den Grundrechenarten, wenn der Schüler beim Rechnen sagt, was er denkt? Auf Grund der Lerntheorie Galperins kann man annehmen, dass die Fehler beim lauten Denken abnehmen werden.

4. Wieviel erklärt die Beherrschung der Grundrechenarten und der schriftlichen Rechenverfahren von der Mathematiknote des Schülers?

SCHÜLERINTERVIEWS UND KLASSIFIZIERUNG DER FEHLER

Mathematik ist ein schwieriges Schulfach für manche Schüler. Es ist wichtig zu wissen, wie der Schüler zu einem fehlerhaften Ergebnis gekommen ist. Wenn der Schüler lernschwach ist, muss der Lehrer seine Fehler z.B. mit Hilfe des Schülerinterviews analysieren (siehe Krutetskii 1976, Lankford 1974).

Um über die Arbeit des Schülers so ausführliche Information wie möglich zu bekommen, muss man irgendein Klassifizierungssystem haben, nach welchem die Ergebnisse der Schüler codiert werden können. Es gibt Forschungen, in denen versucht wird, alle Typen der Schülerfehler zu klassifizieren (z.B. Ashlock 1972, Reisman 1972, Cox 1975, Gerster 1982). Cox versuchte besonders das

Vorkommen von systematischen Fehlern zu ermitteln. Nach Cox kann ein Fehler systematisch genannt werden, falls er zumindest in drei von fünf Aufgaben vorkommt. Cox grenzt deutlich systematische Fehler von Zufallsfehlern ab. Zufallsfehler beweisen nicht, dass der Schüler konsequent falsch denkt.

Cox klassifizierte die Schülerantworten in fünf verschiedene Kategorien.

1. Systematische Fehler. Der Fehler wird in wenigstens drei von fünf Aufgaben gemacht. Der Fehler soll immer vom gleichen Typ beim schriftlichen Rechenverfahren sein.

2. Zufallsfehler. Der Fehler wird in wenigstens drei von fünf Aufgaben gemacht, aber man kann keinen systematisch falschen Prozess erkennen.

3. Konzentrationsfehler. Der Fehler steckt nur in ein oder zwei Aufgaben. Der Schüler weiss jedoch, wie das schriftliche Rechenverfahren ausgeführt wird.

4. Kein Fehler.

5. Unvollständige Antworten. Der Schüler hat nicht alle fünf Aufgaben bearbeitet.

Nach Cox gibt es die meisten systematischen Fehler bei der Addition (13 %), bei anderen schriftlichen Rechenverfahren liegt der Fehlerprozentsatz bei etwa 5-6 %. Es gibt mehr systematische Fehler in den unteren Klassen.

DURCHFÜHRUNG DER UNTERSUCHUNG

Die Versuchspersonen wurden aus der 7. Klasse einer Schule in Helsinki ausgewählt. Es waren alles Schüler, die den niedrigen Niveaukursus in Mathematik gewählt hatten. Die Anzahl der Ver-

suchspersonen war 19 beim Test der Grundrechenarten. Von ihnen waren 12 Jungen und 7 Mädchen. An dem Interview nahmen fünf Schüler teil, die alle Jungen waren.

Der Test war aus verschiedenen Niveaustufen aufgebaut. Bei der Addition waren es sechs Stufen, bei der Substraktion vier, bei der Multiplikation sechs und bei der Division fünf. Die Testaufgaben waren nach steigenden Niveaustufen, aber nicht unbedingt nach steigender Schwierigkeit geordnet. Der entgültige Test enthielt 40 Aufgaben, 10 Aufgaben aus jeder Grundrechenart. (Die Testaufgaben in verkleinerter Form sind im Anhang zu finden.)

Sammeln des Materials

Der schriftliche Rechentest wurde im Frühjahr 1982 für alle Versuchspersonen gleichzeitig in der Mathematikstunde durchgeführt. Die Schüler hatten insgesamt 60 Minuten Zeit. Für die meisten Schüler reichten die 45 min der regulären Unterrichtsstunde.

Die Interviews

Auf Grund des Tests wurden die fünf Schüler ausgewählt, die die niedrigsten Gesamtpunktzahlen bekommen hatten. Diese Schüler wurden interviewt. Das Interview jedes Schülers wurde persönlich auf Grund der Testergebnisse zusammenstellt. Für jeden Schüler wurde etwa 30 min Zeit benötigt und alle Interviews wurden auf Tonband aufgenommen. Die Schüler wurden gebeten, Aufgaben zu lösen und beim Rechnen laut zu denken. Ihnen wurde empfohlen, alles was ihnen in den Sinn kam, laut zu sagen. Sie wurden auch informiert, dass wir nicht in erster Linie an ihren Ergebnissen interessiert waren, sondern daran, was sie denken, wenn sie rechnen.

DIE ERGEBNISSE

Der Test enthielt 40 Aufgaben, also war die grösstmögliche Punktzahl 40. Die Aufgaben wurden dichotomisch bewertet. Der Mittelwert der ganzen Unterrichtsgruppe war in diesem Test 29.2 und die Standardabweichung 7.0.

Fehleranalyse

Die meisten Fehler traten bei der Division auf: 44 % aller Fehlern. Bei der Addition waren es 14 %, bei der Subtraktion 17 % und bei der Multiplikation 25 %. Bei der Klassifizierung wurde das System von Cox angewendet, wo die Fehler nach systematischen, Zufalls- und Konzentrationsfehlern klassifiziert werden. Eine weitere Klasse war die der nicht bearbeiteten Aufgaben.

systematische Fehler	Zufallsfehler	Konzentrations-Fehler	nicht bearbeitete Aufgaben
16 %	24 %	35 %	24 %
21 %	32 %	46 %	

FIGUR 3.

16 % aller Fehler waren systematische Fehler. Sieben verschiedene Fehlertypen liessen sich dieser Kategorie zuordnen. 24 % waren Zufallsfehler und 35 % Konzentrationsfehler. Wenn auch die nicht bearbeiteten Aufgaben dazu gerechnet werden, betrug ihr Anteil 24 %. Wenn die nicht bearbeiteten Aufgaben ausser Betracht gelassen werden, gab es 21 % systematische Fehler, 32 % Zufallsfehler und 46 % Konzentrationsfehler (Fig. 3).

Wenn man die Fehler nach Rechenarten aufschlüsselt, erhält man folgende Ergebnisse. Bei der Addition waren 46 % aller Fehler systematische Fehler, 25 % Zufallsfehler und 29 % Konzentrationsfehler. Hier traten mehr systematische Fehler auf als bei den anderen Rechenarten. Bei der Subtraktion waren 20 % systematische Fehler, 40 % Zufallsfehler und auch 40 % Konzentrationsfehler. Bei der Multiplikation waren 56 % Konzentrationsfehler. Diese Fehler deuten darauf hin, dass die Schüler nicht das Einmaleins können oder dass sie den Übertrag vergessen. Der Anteil der systematischen Fehler betrug 10 %; 11 % waren Zufallsfehler und 23 % nicht bearbeitete Aufgaben. Bei der Division verursachten die nicht bearbeiteten Aufgaben die meisten fehlerhaften Ausführungen. Sie bildeten 42 % der falsch bearbeiteten Aufgaben. Die schwachen Schüler betrachteten die Division als so schwierig, dass sie nicht einmal den Versuch wagten. Systematische Fehler gab es 9 %; 26 % waren Zufallsfehler und 23 % Konzentrationsfehler (Fig. 4).

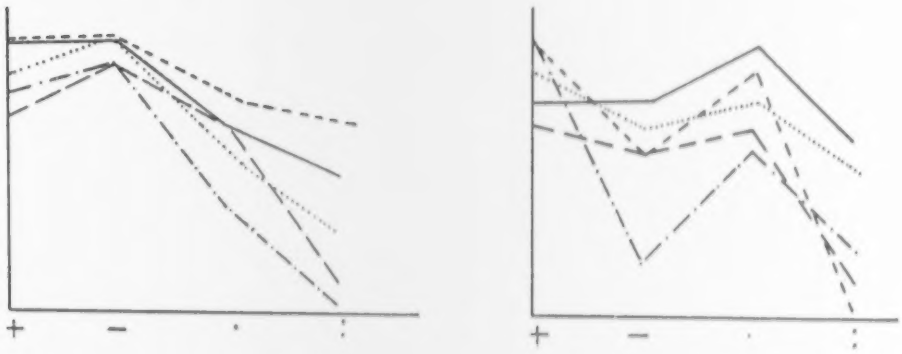
	system. Fehler	Zufalls-fehler	Konzentr.-fehler	nicht bearbeitete Aufgaben
+	46 %	25 %	29 %	-
-	20 %	40 %	40 %	-
.	10 %	11 %	56 %	23 %
:	9 %	26 %	23 %	42 %

FIGUR 4.

Die Leistungsprofile

Auf Grund der Testergebnisse wurde ein Leistungsprofil für jeden Schüler erstellt, damit der Lehrer Schüler mit ähnlichen Leistungsprofilen für dieselbe Förderunterrichtsgruppe auswählen

konnte. In der untersuchten Unterrichtsgruppe wurden zwei Gruppen gefunden, deren Leistungsprofile die gleiche Form hatten (Fig. 5). Für die erste Gruppe ist es charakteristisch, dass das Profil eine Spitze bei der Subtraktion hat, aber bei der Multiplikation abfällt. Das Leistungsprofil der zweiten Gruppe fällt bei der Subtraktion ab, steigt aber bei der Multiplikation wieder an, um bei der Division wieder abzufallen.



FIGUR 5.

Der Test erklärte 26 % der Schulnote in Mathematik. Dieser relativ geringe Anteil scheint plausibel, weil in der Sekundarstufe I mechanisches Rechnen nicht mehr geübt wird.

Interviews

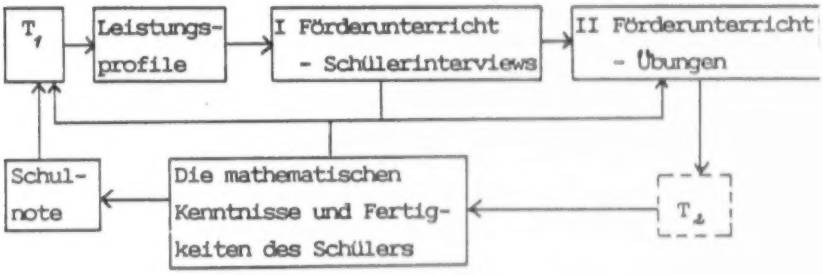
Nach der Fehleranalyse und dem Erstellen der Profile wurden die fünf Schwächsten für das Interview ausgewählt. Der Mittelwert ihrer Punktzahlen war 19.8. Von 29 im schriftlichen Test falsch gerechneten Aufgaben wurden im Interview etwa 2/3 richtig gelöst, als die Schüler "laut" dabei dachten.

Im Interview durften die Schüler bei Bedarf konkrete Gegenstände benutzen (Gummis, Bleistifte u.s.w.), die auf einem daneben-

stehenden Tisch lagen. Alle Schüler brauchten auch ihre Finger beim Rechnen. Bei der Division arbeiteten alle fünf Versuchspersonen noch auf der konkreten Ebene. Mit Hilfe der Gegenstände konnten sie so gerade noch leichte Divisionen rechnen. Eine von fünf Versuchspersonen war bei der Addition offensichtlich auf der Ebene der inneren Sprache. Die anderen Versuchspersonen hätten noch Übung auf der Ebene der äusseren Sprache gebraucht, besonders bei der Subtraktion und Multiplikation.

EIN MODELL FÜR FÖRDERUNTERRICHT

Es ist schwierig, individuellen Unterricht in der Klassensituation zu geben. Der Förderunterricht bietet aber eine Möglichkeit dazu. Hier soll ein Modell dargestellt werden, wie man Förderunterricht orgarisieren kann (siehe Fig. 6).



FIGUR 6.

Auf Grund des (z.B. formativen) Tests T werden die Leistungsprofile der Schüler angefertigt. Mittels deren bildet der Lehrer die Gruppe (oder Gruppen) für den Förderunterricht. Falls man keine gleichförmigen Profile findet, kann man auch einen Schüler allein (wenigstens in Finnland) unterrichten.

In der Phase I des Förderunterrichts versucht man, anhand des Schülerinterviews zu ermitteln, welche Art von Übungen der Schüler benötigt. In der Phase II übt der Schüler z.B. mit

konkretem Material auch noch in der Sekundarstufe I, falls es nötig sein sollte. Der Förderunterricht bietet auch eine Möglichkeit "laut" zu denken. Dies sollte man bei Bedarf unbedingt ermöglichen, weil es scheint, dass einige Schüler nicht einfach direkt übergehen können, im Kopf zu rechnen.

Mittels eines möglichen Tests T kann man kontrollieren, was erreicht wurde. Der gegebene Förderunterricht sollte Einwirkung auf die mathematischen Kenntnisse und Fertigkeiten des Schülers haben; diese haben weiter Einfluss auf die Schulnote und den Erfolg in der nächsten Prüfung.

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Der schriftliche Rechentest

Yhteenlaskuja

1. 1023 + 675	
2. 365 + 834	
3. 432 + 184	4. 1764 + 227
5. 36 + 89	6. 374 + 831
7. 2746 + 2683	8. 15648 + 3927
9. 374 + 837	10. 2476 + 2634

Vähennyslaskuja

1. 387 - 23	2. 567 - 324	
3. 246 - 27	4. 632 - 429	
5. 343 - 54	6. 366 - 177	7. 432 - 156
8. 207 - 39	9. 304 - 177	10. 3054 - 976

Kertolaskuja

1. 3 · 13	2. 4 · 221	
3. 8 · 43	4. 5 · 219	
5. 6 · 329 laske tähän →		
6. 41 · 13 laske tähän →		
7. 36 · 12	8. 37 · 15	9. 39 · 65
10. 29 · 1658 laske tähän →		

Jakolaskuja

1. 72 : 4	2. 423 : 3	9. 975 : 5
4. 834 : 6 laske tähän →		
5. 702 : 3	6. 1024 : 4	
7. 108 : 12 laske tähän →		
8. 221 : 13	9. 609 : 29	10. 9420 : 12

BIOGRAPHICAL NOTES

Gunter Graumann

Professor fur Didaktik der Mathematik an der Universität Bielefeld, Bundesrepublik Deutschland

Tapio Keranto

Associate Professor of Mathematical Subjects Didactics, Doctor of Education, Department of Teacher Training, University of Oulu. Author of mathematics textbooks for the lower level of the comprehensive school.

Erkki Pehkonen

Lecturer of Mathematics Didactics, Doctor of Science, Department of Teacher Training, University of Helsinki. Author of mathematics textbooks and teaching materials for the comprehensive school.

Leila Pehkonen

Class teacher n the lower level of Taivallahti comprehensive school, Helsinki. Author of teaching materials for the lower level of the comprehensive school.

Bernd Zimmermann

Dr., FB Erziehungswissenshaft, Institut 9, Universität Hamburg, Bundesrepublik Deutschland.

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